Math 429 - Representation Theory II

Lie algebras

New concepts will be written in **bold**, and new formulas will be $\boxed{\text{boxed}}$.

Material which you have already encountered in Math 211 and 314 will be marked as such.

Details in the proofs that we purposely leave out of the notes, so that you may work out for yourselves, will be colored in blue. Ask your instructors (in person / on the forum) for help.

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Lecture 1

1.1

A Lie group is a group that happens to also be a manifold, with the two structures being compatible as in Definition 5. The main motivational example is the general linear group of invertible real $n \times n$ matrices, which can be regarded as a subset of $\mathbb{R}^{n^2}$. Before we give the precise definition of Lie groups, recall that a **group** G is a set endowed with

- an identity element $e \in G$,
- an involution $G \to G,\ g \mapsto g^{-1}$,
- an operation $G \times G \to G,\ (g, h) \mapsto gh$ that satisfies associativity.

The structures above must be compatible in the usual ways, that you recall from Math 211. We will now cover some basic differential geometry, and refer you to Math 322 for proofs and details.

Definition 1. *Let $U \subseteq \mathbb{R}^d$ be an open subset. A map*

$$f : U \to \mathbb{R}^n \tag{1}$$

*will be called **smooth** if it is infinitely differentiable (or C^∞), i.e.*

$$f = \Big(f_1(x_1, \dots, x_d), \dots, f_n(x_1, \dots, x_d)\Big)$$

with all partial derivatives of $f_1, \dots, f_n$ well-defined on the set U.

Consider a smooth map (1) and any point $p \in U$. The **derivative** of f at p is defined as

$$f_{*,p} : T_pU \to T_{f(p)}\mathbb{R}^n, \qquad D_pf = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \dots & \frac{\partial f_1}{\partial x_d}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(p) & \dots & \frac{\partial f_n}{\partial x_d}(p) \end{pmatrix} \tag{2}$$

(we think of T_pU as the vector space of tangent vectors to $U \subseteq \mathbb{R}^d$ at p, even though it is naturally identified with $\mathbb{R}^d$). A smooth map f is called an immersion if $f_{*,p}$ is injective at all points $p \in U$.

Definition 2. *A subset $G \subseteq \mathbb{R}^n$ is called a d-dimensional **submanifold** of $\mathbb{R}^n$ if for every $g \in G$ there exists an open subset $U \subseteq \mathbb{R}^d$ and an embedding* [1]

$$\phi : U \to \mathbb{R}^n$$

such that $g \in \mathrm{Im}\ \phi \subseteq G$.

In other words, a submanifold is a subset of $\mathbb{R}^n$ where every point has a neighborhood that can be identified with an open subset $U \subseteq \mathbb{R}^d$ via the embedding ϕ, the latter being called a **chart** at g. For example, a sphere is a 2-dimensional submanifold of $\mathbb{R}^3$, as indicated in the following picture.

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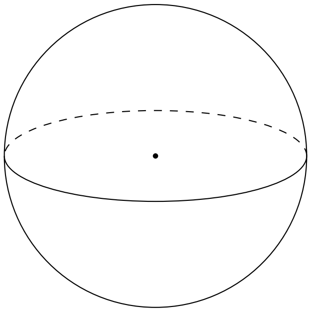
[1] An embedding is an immersion which is a homeomorphism onto its image; the reason for the latter requirement is to preclude examples such as everywhere dense curves in $\mathbb{R}^2$ from being considered submanifolds.



In what follows, we will refer to submanifolds of various $\mathbb{R}^n$ as simply **manifolds**. In Math 322, you learned how to define manifolds abstractly, but we will not need this in the present course.

Definition 3. *Given manifolds G, G' of dimensions d, d' (respectively), a map*

$$f : G \to G' \tag{3}$$

is called ***smooth*** *if for every $g \in G$ there is a chart $\phi : U \to G$ at g and a chart $\phi' : U' \to G'$ at $f(g)$ such that $\phi'^{-1} \circ f \circ \phi$ is a smooth map from an open subset of $\mathbb{R}^d$ to an open subset of $\mathbb{R}^{d'}$.*

Because charts of d-dimensional submanifolds $G \subseteq \mathbb{R}^n$ are embeddings, we have that

$$T_g G = \text{Im } \phi_{*,\phi^{-1}(g)} \subseteq T_g \mathbb{R}^n \tag{4}$$

has dimension d, for any chart ϕ at any point $g \in G$. We call $T_g G$ the **tangent space** to G at g. Any smooth map $f : G \to G'$ induces a linear transformation on tangent spaces

$$f_{*,g} : T_g G \to T_{f(g)} G' \tag{5}$$

for any $g \in G$.

Definition 4. *A smooth map $f : G \to G'$ is called*

- *an* ***immersion*** *if $f_{*,g}$ is injective at every point $g \in G$;*
- *an* ***embedding*** *if it is both an immersion and a homeomorphism onto its image;*
- *a* ***diffeomorphism*** *if it is a bijective embedding (in this case, f^{-1} will also be smooth).*

Definition 5. *A* ***Lie group*** *is a set which has both a structure of group and of a manifold, such that the inverse $G \to G$ and the operation $G \times G \to G$ are smooth maps.*

1.2

Here are some important examples of Lie groups; we let you check that they satisfy Definition 5.

- the **general linear group** $\boxed{GL_n(\mathbb{R}) = \left\{ A \in \text{Mat}_{n\times n}(\mathbb{R}) \middle| \det(A) \neq 0 \right\}}$

- the **special linear group** $\boxed{SL_n(\mathbb{R}) = \left\{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \middle| \det(A) = 1\right\}}$
- the **orthogonal group** $\boxed{O_n(\mathbb{R}) = \left\{A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \middle| AA^T = I_n\right\}}$
- the **special orthogonal group** $\boxed{SO_n(\mathbb{R}) = O_n(\mathbb{R}) \cap SL_n(\mathbb{R})}$
- the **symplectic group** $\boxed{Sp_{2n}(\mathbb{R}) = \left\{A \in \mathrm{Mat}_{2n\times 2n}(\mathbb{R}) \middle| AJ_-A^T = J_-\right\}}$, where

$$J_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

- $\mathbb{R}^n$ with component-wise addition, and $(\mathbb{R}^*)^n$ with component-wise multiplication.

Definition 6. *A **complex Lie group** is defined just like in Subsection 1.1, but replacing the terms* ($\mathbb{R}$, *smooth map*, *manifold*) *by* ($\mathbb{C}$, *holomorphic map*, *complex manifold*) *everywhere; we may use the terms "real manifold" and "real Lie group" to refer to Definitions 2 and 5 when contrasting them with their complex analogues.*

Examples of complex Lie groups include

$$GL_n(\mathbb{C}),\ SL_n(\mathbb{C}),\ O_n(\mathbb{C}),\ SO_n(\mathbb{C}),\ Sp_{2n}(\mathbb{C})$$

defined as above, but with complex instead of real coefficients. In order to better integrate the formulas for complex orthogonal and symplectic groups, we will actually redefine the former as

$$\boxed{O_n(\mathbb{C}) = \left\{A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \middle| AJ_+A^T = J_+\right\}} \tag{7}$$

and $SO_n(\mathbb{C}) = O_n(\mathbb{C}) \cap SL_n(\mathbb{C})$, where

$$J_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

(intrinsically, the orthogonal group is the set of linear transformations which preserve a certain symmetric bilinear form; over the complex numbers, the choices $\langle e_i, e_j\rangle = \delta_{ij}$ and $\langle e_i, e_j\rangle = \delta_{i,n+1-j}$ give equivalent definitions). It is easy to see that any complex Lie group is a real Lie group, but the converse is not true. For example, the **unitary group**

$$\boxed{U(n) = \left\{A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \middle| A\bar{A}^T = I_n\right\}} \tag{9}$$

and the **special unitary group**

$$\boxed{SU(n) = U(n) \cap SL_n(\mathbb{C})}$$

are real Lie groups and not complex Lie groups, despite the fact that they are defined as subsets of the set of complex matrices. For example, $U(1) = S^1 \subset \mathbb{C}^*$ is a Lie group with the operation given by rotation, and there is no reasonable sense in which a circle can be a complex manifold.

Remark. *The world of real manifolds is richer in Lie groups than the world of complex manifolds. For instance, recall that the orthogonal group can be thought of as the set of linear transformations which preserve the Euclidean inner product. One can consider the generalized orthogonal groups, defined as the set of linear transformations which preserve a bilinear form of signature $(k, n-k)$:*

$$O_{k,n-k}(\mathbb{R}) = \left\{ A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \middle| A \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & -I_{n-k} \end{array} \right) A^T = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & -I_{n-k} \end{array} \right) \right\}$$

These groups are all non-isomorphic (except when $k \leftrightarrow n-k$), but if we replace $\mathbb{R}$ by $\mathbb{C}$ above, then they all become isomorphic to $O_n(\mathbb{C})$. The flip side of this is that complex Lie groups are in general better behaved than real Lie groups.

1.3

We will now give an intrinsic definition of the tangent spaces (4) of a manifold.

Definition 7. *Let g be a point in a manifold G. A linear functional v on the real vector space*

$$\left\{ \textit{smooth functions } G \xrightarrow{\lambda} \mathbb{R} \right\}$$

*is called a **derivation** at g if it satisfies the Leibniz rule*

$$v(\lambda\mu) = v(\lambda)\mu(g) + \lambda(g)v(\mu) \tag{10}$$

for all smooth functions $\lambda, \mu : G \to \mathbb{R}$.

With this in mind, we have an identification

$$T_g G = \left\{ \text{derivations at } g \right\} \tag{11}$$

which is given by the following assignment in a chart $\phi : U \to G$:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \mapsto \left(G \xrightarrow{\lambda} \mathbb{R} \quad \rightsquigarrow \quad \sum_{i=1}^{d} \alpha_i \frac{\partial(\lambda \circ \phi)}{\partial x_i}(\phi^{-1}(g)) \right) \tag{12}$$

It is easy to show that any smooth map $f : G \to G'$ induces a linear transformation $T_g G \to T_{f(g)} G'$.

Definition 8. *A **vector field** on a manifold G refers to a choice $\boldsymbol{v} = \{v_g \in T_gG\}_{g\in G}$ of tangent vectors that varies smoothly with g. Intrinsically, a vector field is a derivation*

$$\boldsymbol{v} : \Big\{\textit{smooth functions } G \to \mathbb{R}\Big\} \to \Big\{\textit{smooth functions } G \to \mathbb{R}\Big\} \tag{13}$$

i.e. it satisfies the Leibniz rule in the following form

$$\boldsymbol{v}(\lambda\mu) = \boldsymbol{v}(\lambda)\mu + \lambda\boldsymbol{v}(\mu) \tag{14}$$

Any smooth map $f : G \to G'$ induces a linear transformation between the sets of vector fields [2].

If G is a Lie group, then we have left multiplication maps for each $g \in G$

$$G \xrightarrow{h\mapsto gh} G$$

which are smooth. Therefore, they induce linear isomorphisms between tangent spaces

$$T_hG \xrightarrow{\text{left multiplication by } g} T_{gh}G$$

Thus, we have a notion of **left invariant** vector field on G, namely a vector field which is preserved by the left multiplication maps above. Since a left invariant vector field is completely determined by its value at the identity, we conclude that

$$\boxed{T_eG \cong \Big\{\text{left invariant vector fields on } G\Big\}} \tag{16}$$

1.4

It is customary to write $\text{Lie}(G) = T_eG$. Although a priori just a vector space, $\text{Lie}(G)$ can be endowed with an extra structure called a **Lie bracket**. The key result is the following.

Proposition 1. *If $\boldsymbol{v}$ and $\boldsymbol{w}$ are two vector fields (i.e. derivations* (13)*), then so is the commutator*

$$\boxed{[\boldsymbol{v}, \boldsymbol{w}]\,(\lambda) = \boldsymbol{v}(\boldsymbol{w}(\lambda)) - \boldsymbol{w}(\boldsymbol{v}(\lambda))} \tag{17}$$

for all smooth $G \xrightarrow{\lambda} \mathbb{R}$. Moreover, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are left invariant vector fields on G, then so is $[\boldsymbol{v}, \boldsymbol{w}]$.

Proof. Explicitly, for any smooth functions λ and μ, we have

$$\begin{aligned}
&[\boldsymbol{v}, \boldsymbol{w}](\lambda\mu) = \boldsymbol{v}(\boldsymbol{w}(\lambda\mu)) - \boldsymbol{w}(\boldsymbol{v}(\lambda\mu)) = \boldsymbol{v}(\boldsymbol{w}(\lambda)\mu + \lambda\boldsymbol{w}(\mu)) - \boldsymbol{w}(\boldsymbol{v}(\lambda)\mu + \lambda\boldsymbol{v}(\mu)) = \\
&= \boldsymbol{v}(\boldsymbol{w}(\lambda))\mu + \boldsymbol{w}(\lambda)\boldsymbol{v}(\mu) + \boldsymbol{v}(\lambda)\boldsymbol{w}(\mu) + \lambda\boldsymbol{v}(\boldsymbol{w}(\mu)) - \boldsymbol{w}(\boldsymbol{v}(\lambda))\mu - \boldsymbol{v}(\lambda)\boldsymbol{w}(\mu) \\
&- \boldsymbol{w}(\lambda)\boldsymbol{v}(\mu) - \lambda\boldsymbol{w}(\boldsymbol{v}(\mu)) = \left(\boldsymbol{v}(\boldsymbol{w}(\lambda)) - \boldsymbol{w}(\boldsymbol{v}(\lambda))\right)\mu + \lambda\left(\boldsymbol{v}(\boldsymbol{w}(\mu)) - \boldsymbol{w}(\boldsymbol{v}(\mu))\right)
\end{aligned}$$

Finally, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are left invariant (i.e. preserved by the automorphisms of left multiplication), then so are the compositions $\boldsymbol{v} \circ \boldsymbol{w}$ and $\boldsymbol{w} \circ \boldsymbol{v}$ of these vector fields, hence so is $[\boldsymbol{v}, \boldsymbol{w}]$. ☐

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[2] If we take a chart $\phi : U \to G$, then any vector field locally takes the form

$$G \xrightarrow{\lambda} \mathbb{R} \quad \rightsquigarrow \quad \text{function locally given by } \phi^{-1}\left(\sum_{i=1}^{d} \alpha_i \frac{\partial(\lambda\circ\phi)}{\partial x_i}\right) \tag{15}$$

for various smooth functions $\alpha_1, \dots, \alpha_d$ on U.

The commutator of vector fields is actually a very explicit operation. In local coordinates $x_1, \dots, x_d$ given by choosing a chart $\phi : U \to G$, we may write vector fields as

$$\boldsymbol{v} = \alpha_1 \frac{\partial}{\partial x_1} + \dots + \alpha_d \frac{\partial}{\partial x_d} \quad \text{and} \quad \boldsymbol{w} = \beta_1 \frac{\partial}{\partial x_1} + \dots + \beta_d \frac{\partial}{\partial x_d}$$

for various smooth functions $\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d$ on U. Then we have

$$[\boldsymbol{v}, \boldsymbol{w}] = \sum_{i=1}^{d} \sum_{j=1}^{d} \left(\alpha_i \frac{\partial \beta_j}{\partial x_i} - \beta_i \frac{\partial \alpha_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} \tag{18}$$

It is easy to see that the operation (17) satisfies the following properties:

- anti-symmetry: $[\boldsymbol{v}, \boldsymbol{w}] = -[\boldsymbol{w}, \boldsymbol{v}]$;
- the Jacobi identity $[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] + [\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]] + [\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]] = 0$.

We conclude that the vector space $\text{Lie}(G)$ of (16) is endowed with an operation

$$\text{Lie}(G) \times \text{Lie}(G) \xrightarrow{[\cdot,\cdot]} \text{Lie}(G)$$

satisfying anti-symmetry and the Jacobi identity. If we regard $\text{Lie}(G)$ as the tangent space at $e \in G$, this operation can be described as follows: take any two tangent vectors, extend them uniquely to left-invariant vector fields on G, then take the commutator of the vector fields in question as per (17), and then restrict the corresponding vector field back to $e \in G$.

1.5

The following definition is an abstract version of the discussion in the previous Subsection, which actually makes sense over any ground field $\mathbb{K}$.

Definition 9. *A* ***Lie algebra*** $\mathfrak{g}$ *is a* $\mathbb{K}$*-vector space endowed with a* ***Lie bracket***

$$\mathfrak{g} \times \mathfrak{g} \xrightarrow{[\cdot,\cdot]} \mathfrak{g} \tag{19}$$

which is $\mathbb{K}$*-bilinear in both arguments and satisfies*

- *anti-symmetry:* $[x, y] = -[y, x]$*, and*
- *the Jacobi identity*
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{20}$$

All Lie algebras studied in this course will be finite-dimensional.

Thus, the discussion in the last paragraph of the previous Subsection shows that $\text{Lie}(G)$ has the structure of a Lie algebra, for all Lie groups G.

Example 1. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let us work out the Lie algebra structure on*

$$\mathfrak{gl}_{n,\mathbb{K}} = \mathrm{Lie}(GL_n(\mathbb{K}))$$

Because $GL_n(\mathbb{K})$ is an open subset of the $\mathbb{K}$-vector space of all $n \times n$ matrices, its tangent spaces are all naturally identified with the vector space in question, so we have an identification

$$\mathfrak{gl}_{n,\mathbb{K}} = \mathrm{Mat}_{n\times n}(\mathbb{K})$$

Show that the left invariant vector field corresponding to $X \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ is

$$(gX)_{g\in GL_n(\mathbb{K})}$$

Given $X, Y \in \mathrm{Mat}_{n\times n}(\mathbb{K})$, calculate the commutator of the corresponding vector fields by (18)

$$\left[(gX)_{g\in GL_n(\mathbb{K})}, (gY)_{g\in GL_n(\mathbb{K})}\right] = \sum_{1\le i,j,k,\ell\le n} g_{ik}\left(X_{k\ell}Y_{\ell j} - Y_{k\ell}X_{\ell j}\right)\frac{\partial}{\partial E_{ij}}$$

where we write matrices $g = \sum_{1\le i,j\le n} g_{ij}E_{ij}$ etc, in terms of their coefficients in the standard basis (thus E_{ij} is the $n \times n$ matrix with entry 1 on position (i,j) and 0 everywhere else). Restricting the above equality to the identity $\{g_{ik} = \delta_{ik}\}_{i,k\in\{1,\dots,n\}}$ shows that the Lie bracket on $\mathfrak{gl}_{n,\mathbb{K}}$ is given by

$$\mathfrak{gl}_{n,\mathbb{K}} \times \mathfrak{gl}_{n,\mathbb{K}} \xrightarrow{[\cdot,\cdot]} \mathfrak{gl}_{n,\mathbb{K}}, \qquad \boxed{[X,Y] = XY - YX}$$

1.6

Let us now give examples of Lie algebras beside $\mathrm{Lie}(GL_n(\mathbb{K})) = \mathfrak{gl}_{n,\mathbb{K}}$; we still let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Firstly, note that it's quite easy to determine the Lie algebras of matrix groups (i.e. subgroups of $GL_n(\mathbb{K})$ cut out by polynomial equations). For instance you will show in the exercise session that

$$\mathrm{Lie}(SL_n(\mathbb{K})) = \mathfrak{sl}_{n,\mathbb{K}} = \left\{X \in \mathrm{Mat}_{n\times n}(\mathbb{K})\middle|\mathrm{tr}(X) = 0\right\}$$

$$\mathrm{Lie}(O_n(\mathbb{K})) = \mathfrak{o}_{n,\mathbb{K}} = \left\{X \in \mathrm{Mat}_{n\times n}(\mathbb{K})\middle|XJ_+ + J_+X^T = 0\right\} = \mathfrak{so}_{n,\mathbb{K}} = \mathrm{Lie}(SO_n(\mathbb{K}))$$

$$\mathrm{Lie}(Sp_{2n}(\mathbb{K})) = \mathfrak{sp}_{2n,\mathbb{K}} = \left\{X \in \mathrm{Mat}_{2n\times 2n}(\mathbb{K})\middle|XJ_- + J_-X^T = 0\right\}$$

$$\mathrm{Lie}(U(n)) = \mathfrak{u}(n) = \left\{X \in \mathrm{Mat}_{n\times n}(\mathbb{C})\middle|X + \bar{X}^T = 0\right\}$$

$$\mathrm{Lie}(SU(n)) = \mathfrak{su}(n) = \left\{X \in \mathrm{Mat}_{n\times n}(\mathbb{C})\middle|X + \bar{X}^T = 0 \text{ and } \mathrm{tr}(X) = 0\right\}$$

[3] Secondly, there are many more Lie algebras out there than Lie groups. For one thing, Lie algebras can be defined over any field (including characteristic p) and they may be infinite-dimensional, neither of which situation is compatible with being the tangent space of a Lie group. For example, the **Virasoro algebra** is an important infinite-dimensional Lie algebra

$$\mathrm{Vir} = \mathbb{C}c \oplus \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n, \qquad [c, L_n] = 0, \qquad [L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{c(m^3-m)}{12}$$

which is the fundamental object in conformal field theory. We will not study Lie algebras of infinite dimension or positive characteristic in this course, but they are very rich subjects.

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[3] Note that for $\mathfrak{o}_{n,\mathbb{K}}$ and $\mathfrak{so}_{n,\mathbb{K}}$, we are using the convention for the orthogonal group in (7).

Lecture 2

2.1

Many of the usual constructions for groups apply to Lie groups as well, but we must be careful to make sure the manifold structure is preserved. For example, a **Lie group homomorphism**

$$f : G \to G' \tag{21}$$

is required to be both a group homomorphism and a smooth map. Similarly, a linear function

$$f : \mathfrak{g} \to \mathfrak{g}'$$

is called a **Lie algebra homomorphism** if it preserves the Lie bracket:

$$\boxed{f([x,y]) = [f(x), f(y)]} \tag{22}$$

$\forall x, y \in \mathfrak{g}$, where the LHS involves the Lie bracket in $\mathfrak{g}$ and the RHS involves the Lie bracket in $\mathfrak{g}'$.

Proposition 2. *If $f : G \to G'$ is a Lie group homomorphism, the derivative*

$$f_* = f_{*,e} : \mathfrak{g} \to \mathfrak{g}'$$

is a Lie algebra homomorphism, where $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}' = \mathrm{Lie}(G')$.

Proof. The linear map on vector fields induced by f takes left-invariant vector fields to left-invariant vector fields. The fact that this map satisfies (22) is automatic. ☐

2.2

We will now review the basic representation theory of Lie groups, generalizing the situation for usual groups that you already encountered in Math 211 or 314. We say that a real/complex Lie group G **acts** on a real/complex manifold M, denoted by

$$\boxed{G \curvearrowright M} \tag{23}$$

if there exists a smooth/holomorphic map

$$G \times M \to M, \qquad (g, m) \mapsto \Phi_g(m) = g \cdot m$$

that simultaneously satisfies the usual properties from group theory

$$\Phi_{gg'} = \Phi_g \circ \Phi_{g'} \qquad \text{and} \qquad \Phi_e = \mathrm{Id}_M \tag{24}$$

and is a smooth/holomorphic map of real/complex manifolds.

Example 2. *Because the groups $SL_n(\mathbb{K}), O_n(\mathbb{K}), SO_n(\mathbb{K})$ etc are subgroups of $GL_n(\mathbb{K})$, they naturally act on $\mathbb{K}^n$. More interesting actions can be obtained by observing that various subsets of n-dimensional space are preserved by the aforementioned actions, for instance*

$$O_n(\mathbb{R}) \curvearrowright S^{n-1} = \left\{ x_1^2 + \cdots + x_n^2 = 1 \right\} \subset \mathbb{R}^n$$
$$U(n) \curvearrowright S^{2n-1} = \left\{ |z_1|^2 + \cdots + |z_n|^2 = 1 \right\} \subset \mathbb{C}^n$$

As in the usual case of group theory, we have the actions of a Lie group G on itself

$$\begin{aligned} &\text{left action } g \cdot h = gh \\ &\text{right action } g \cdot h = hg^{-1} \\ &\text{adjoint action } g \cdot h = ghg^{-1} \end{aligned}$$

The **orbits** of an action $G \curvearrowright M$ are the sets

$$Gm = \Big\{ g \cdot m \Big| g \in G \Big\}$$

as m runs over M. While the left and right actions $G \curvearrowright G$ have a single orbit (in other words, they are transitive), the orbits of the adjoint action are just the conjugacy classes of G.

2.3

If a Lie group G acts on a (real or complex) vector space V in such a way that all the action maps

$$\Phi_g : V \to V$$

are linear transformations, then we say that V is a **representation of** G [4]. This can be rephrased in terms of the Lie group (with operation given by composition)

$$GL(V) = \Big\{ \text{invertible linear transformations } V \to V \Big\} \tag{25}$$

in that giving a representation $G \curvearrowright V$ is the same thing as giving a Lie group homomorphism

$$\boxed{G \to GL(V)}, \quad g \mapsto \Big(\Phi_g : V \to V \Big) \tag{26}$$

Taking the derivative of (26) at the identity gives us

$$\boxed{\mathfrak{g} \to \mathfrak{gl}(V)}, \quad x \mapsto \Big(\phi_x : V \to V \Big) \tag{27}$$

where we write $\mathfrak{g} = \text{Lie}(G)$ and define

$$\mathfrak{gl}(V) = \text{End}(V) := \Big\{ \text{linear transformations } V \to V \Big\} \tag{28}$$

Note that (28) is a vector space with respect to addition and a Lie algebra with respect to commutator, and it coincides with $\text{Lie}(GL(V))$. The following statement is an immediate consequence of (27) being a Lie algebra homomorphism, which follows from Proposition 2.

Proposition 3. *The assignment* (27) *yields a* ***Lie algebra representation****, i.e. an assignment*

$$\Big\{ \phi_x : V \to V \Big\}_{x \in \mathfrak{g}}$$

of linear transformations (which depend linearly on x*) such that*

$$\phi_{[x,y]} = \phi_x \circ \phi_y - \phi_y \circ \phi_x \tag{29}$$

for all $x, y \in \mathfrak{g}$.

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[4] Note that we allow complex representations V of real Lie groups G; in this case, each Φ_g is required to be $\mathbb{C}$-linear, but the action map $G \times V \to V$ is only required to be smooth.

2.4

The adjoint action $\mathrm{Ad}_g(h) = ghg^{-1}$

$$G \curvearrowright G, \quad \mathrm{Ad}_g : G \to G \tag{30}$$

does not constitute a representation because G is not a vector space, but its derivative

$$\boxed{G \curvearrowright \mathfrak{g}, \quad \mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}} \tag{31}$$

is a representation, according to the following.

Proposition 4. *Formula* (31) *is a Lie group representation, and its derivative*

$$\boxed{\mathfrak{g} \curvearrowright \mathfrak{g}, \quad \mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}} \tag{32}$$

is a Lie algebra representation. Explicitly,

$$\boxed{\mathrm{ad}_x(y) = [x, y]} \tag{33}$$

for all $x, y \in \mathfrak{g}$. Both (31) *and* (32) *are called the* ***adjoint representation****.*

Proof. The fact that (31) is a Lie group representation is immediate, as $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a linear function (as are all derivatives of smooth maps) which inherits properties (24) from $\mathrm{Ad}_g : G \to G$. Therefore, its derivative (32) is a Lie algebra representation by Proposition 2. Formula (33) is best proved by an alternative description of the Lie bracket involving one-parameter subgroups (to be covered in the exercise session), but let us give an explicit computation for $\mathfrak{g} = \mathfrak{gl}_{n,\mathbb{K}}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We have

$$\mathrm{Ad}_g(h) = ghg^{-1}$$

for all $g, h \in GL_n(\mathbb{K})$. Letting $h = 1 + tY$ for an infinitesimal t and any $n \times n$ matrix Y (this is reasonable, since we are identifying the tangent space at $e \in GL_n(\mathbb{K})$ with the ambient space of all matrices), we see that the adjoint representation is given by

$$\mathrm{Ad}_g(Y) = gYg^{-1}$$

for all $g \in GL_n(\mathbb{K})$, $Y \in \mathfrak{gl}_{n,\mathbb{K}}$. To differentiate the formula above, let $g(t) = 1 + tX$ for infinitesimal t. Then

$$g(t)^{-1} = 1 - tX + t^2X^2 - \dots$$

and so

$$\mathrm{ad}_X(Y) = \lim_{t \to 0} \frac{\mathrm{Ad}_{g(t)}(Y) - Y}{t} = \lim_{t \to 0} \frac{(1 + tX)Y(1 - tX + \dots) - Y}{t} = XY - YX$$

which coincides with the Lie bracket of $\mathfrak{gl}_{n,\mathbb{K}}$. ☐

Example 3. *For any vector space V, we have a tautological representation*

$$GL(V) \curvearrowright V$$

We may extend this action naturally to any tensor product of symmetric and exterior powers of V

$$GL(V) \curvearrowright \cdots \otimes S^k V \otimes \wedge^\ell V \otimes \cdots$$

*You have seen at the very end of Math 314 that these representations come into play in **Schur-Weyl duality**. This is a statement that for any $n \in \mathbb{N}$, we have a decomposition*

$$\underbrace{V \otimes \cdots \otimes V}_{n \text{ factors}} = \bigoplus_{\text{partition } \lambda} L(\lambda) \otimes S_\lambda$$

of representations of $GL(V) \times S_n$ (the symmetric group permutes the factors in the LHS), where in the RHS we write S_λ for the irreducible Specht modules of S_n, and $L(\lambda)$ for the irreducible representations of $GL(V)$. We will characterize the latter in more detail in Lectures 13 and 14.

2.5

Many of the basic notions from Math 314 apply to Lie groups as they did to usual groups. Given representations $G \curvearrowright V$ and $G \curvearrowright W$ (determined by collections $\{\Phi_g : V \to V\}_{g \in G}$ and $\{\Psi_g : W \to W\}_{g \in G}$ of linear transformations, respectively) a **G-intertwiner** is a linear transformation

$$f : V \longrightarrow W$$

such that the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\Phi_g"'] & W \arrow[d, "\Psi_g"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

for all $g \in G$. If we write $\Phi_g(v) = g \cdot v$ and $\Psi_g(w) = g \cdot w$ for all $v \in V$ and $w \in W$, then the property of being a G-intertwiner is equivalent to $f(g \cdot v) = g \cdot f(v)$ for all $g \in G, v \in V$. If a G-intertwiner is moreover bijective, then we call it an **isomorphism**. Recall that a subset of a vector space is called a subspace if and only if it is preserved under addition of vectors and scalar multiplication. If we have a representation $G \curvearrowright V$, then a subspace $W \subseteq V$ is called a **subrepresentation** if

$$\Phi_g(W) \subseteq W$$

for all $g \in G$. Moreover, in this case there is an induced **quotient representation**

$$G \curvearrowright V/W$$

Given representations V and W of G, we can make their direct sums, tensor products and duals into G-representations via

$$G \curvearrowright V \oplus W, \quad g \cdot (v, w) = (g(v), g(w)) \tag{34}$$

$$G \curvearrowright V \otimes W, \quad g \cdot (v \otimes w) = g(v) \otimes g(w) \tag{35}$$

$$G \curvearrowright V^\vee, \quad (g \cdot \lambda)(v) = \lambda(g^{-1} \cdot v) \tag{36}$$

The natural analogues of all notions above (intertwiners, isomorphisms, sub-and-quotient representations) apply equally well to Lie algebras as to Lie groups. The only difference lies in formulas (34), (35), (36), which must be modified in the case of Lie algebra representations to

$$\mathfrak{g} \curvearrowright V \oplus W, \quad x \cdot (v, w) = (x(v), x(w)) \tag{37}$$

$$\mathfrak{g} \curvearrowright V \otimes W, \quad x \cdot (v \otimes w) = x(v) \otimes w + v \otimes x(w) \tag{38}$$

$$\mathfrak{g} \curvearrowright V^\vee, \quad (x \cdot \psi)(v) = -\psi(x(v)) \tag{39}$$

(to get from (34), (35), (36) to (37), (38), (39) let $g = \exp(tx)$ and calculate the derivative at $t = 0$).

2.6

We will state the following basic facts for representations of (real or complex) Lie groups G, but they apply equally well for representations of Lie algebras $\mathfrak{g}$ (over any field).

Definition 10. *A representation $G \curvearrowright V$ is called **irreducible** if it does not have any proper subrepresentations (i.e. no subrepresentations other than 0 or V).*

One of the main tools in representation theory is the following result, known as **Schur's lemma**.

Lemma 1. *Suppose we have a G-intertwiner $f : V \to W$ between two representations of G, which is not identically 0. If V is irreducible, then f is injective. If W is irreducible, then f is surjective.*

As an immediate corollary of Lemma 1, any non-zero intertwiner between two irreducible representations must be an isomorphism. All of the above is the same for Lie groups as it was for finite groups, but some things do not generalize so easily. An example of this is Maschke's theorem, which says that any complex finite-dimensional representation of a finite group $G \curvearrowright V$ has the property that any subrepresentation $W \subseteq V$ has a complement

$$V \cong W \oplus W' \tag{40}$$

such that W' is also a subrepresentation of V (an important consequence of this is that finite-dimensional complex representations of finite groups are completely reducible, i.e. isomorphic to direct sums of irreducible representations). This result completely fails for Lie groups in general. For example, consider the action of $\mathbb{C}$ (a Lie group with respect to addition) on $V = \mathbb{C}^2$ via

$$x \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{41}$$

The subspace $W = \{b = 0\}$ is a one-dimensional subrepresentation of V, but because it is the unique such subrepresentation, it is impossible to find a decomposition (40). However, compact Lie groups and unitary representations will give us a setting in which we can salvage these results. We will study these (and the corresponding Lie algebras) in the next Lecture.

Lecture 3

3.1

You have already seen a version of what follows in Math 314, but we include it for review. Recall that a Hilbert space is a $\mathbb{C}$-vector space V endowed with an inner product

$$V \otimes V \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{C}$$

which is linear in the first argument, satisfies $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$ for all $v_1, v_2 \in V$, and $\langle v, v \rangle \in \mathbb{R}_{>0}$ if $v \neq 0$. A linear transformation $f : V \to V$ is called **unitary** if it preserves the inner product

$$\Big\langle f(v_1), f(v_2) \Big\rangle = \Big\langle v_1, v_2 \Big\rangle$$

If $V = \mathbb{C}^n$ with the standard inner product

$$\Big\langle (z_1, \dots, z_n), (w_1, \dots, w_n) \Big\rangle = z_1 \overline{w}_1 + \dots + z_n \overline{w}_n \tag{42}$$

then a unitary linear transformation is simply given by a unitary matrix A as in (9), i.e. $f(v) = Av$.

Definition 11. *If V is a Hilbert space, then a representation $G \curvearrowright V$ is called **unitary** if all the action maps $\Phi_g : V \to V$ are unitary linear transformations.*

Unitary representations satisfy Maschke's theorem. Indeed, given a subrepresentation $W \subseteq V$ of a finite-dimensional unitary representation $G \curvearrowright V$, one defines

$$W' = \Big\{ v \in V \Big| \langle v, W \rangle = 0 \Big\}$$

The fact that all the Φ_g are unitary (i.e. preserve the inner product) implies that all the Φ_g preserve W' (i.e. W' is a subrepresentation), while the fact that $V = W \oplus W'$ follows from bilinearity and the fact that $\langle v, v \rangle \neq 0$ if $v \neq 0$. After repeated applications of Maschke's theorem, one concludes that finite-dimensional unitary representations are **completely reducible**, i.e.

$$V \cong V_1 \oplus \dots \oplus V_k \tag{43}$$

where $V_1, \dots, V_k$ are irreducible representations.

3.2

A complex representation V of a Lie group G is called **unitarizable** if it admits an inner product with respect to which it is a unitary representation. As we saw in the previous Subsection, unitarizable representations satisfy Maschke's theorem and complete reducibility. The following discussion provides a large source of examples of unitarizable representations.

Definition 12. *A real Lie group is called **compact** if it is compact as a topological space.*

All Lie groups we will encounter in this course will be matrix groups, and their compactness is equivalent (by the Heine-Borel theorem) with being closed and bounded. Thus, we see that $SL_n(\mathbb{K}), SO_n(\mathbb{K}), Sp_{2n}(\mathbb{K}), \dots$ for $n \geq 2$ are not compact for any $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, since one can easily cook up a matrix in each of these Lie groups that has top-left entry of arbitrarily large absolute value. However, the unitary groups $U(n)$ and $SU(n)$ are compact because

- they are closed, as the equations $\bar{A}^T A = I_n$ and $\det(A) = 1$ are polynomial in the real and imaginary parts of the entries $\{a_{ij}\}_{1 \leq i,j \leq n}$ of a matrix A,
- they are bounded, as $\sum_{1 \leq i,j \leq n} |a_{ij}|^2 = \mathrm{tr}(\bar{A}^T A) = n$.

Proposition 5. *Any complex representation V of a compact Lie group G is unitarizable.*

Proof. Since $V \cong \mathbb{C}^n$ for some n, we can consider the standard inner product

$$V \otimes V \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{C}$$

given by formula (42). We may define another inner product by "averaging" the one above over the compact Lie group G

$$\Big\langle v_1, v_2 \Big\rangle^{\text{avg}} = \int_G \Big\langle \Phi_g(v_1), \Phi_g(v_2) \Big\rangle dg \tag{44}$$

where dg is known as a **Haar measure** on G (i.e. a measure which is invariant under the right action, i.e. $dg = d(gh^{-1})$ for all $h \in G$; its existence is beyond the scope of our course). The compactness of G means that the formula above is well-defined, and the fact that it determines an inner product is a straightforward check of the axioms, which we recommend you do. To show that V with the inner product (44) is a unitary representation, choose any $h \in G$ and note that

$$\Big\langle \Phi_h(v_1), \Phi_h(v_2) \Big\rangle^{\text{avg}} = \int_G \Big\langle \Phi_{gh}(v_1), \Phi_{gh}(v_2) \Big\rangle dg = \int_G \Big\langle \Phi_g(v_1), \Phi_g(v_2) \Big\rangle d(gh^{-1})$$

The RHS of the above is equal to the RHS of (44) precisely because dg is a Haar measure. ☐

3.3

As we have seen in the previous Subsections, the representation theory of compact Lie groups is simpler than that of arbitrary Lie groups. However, the two can be related by a procedure known as "Weyl's unitary trick". The following is an important result, which we will not prove.

Theorem 1. *Any Lie group has a maximal compact subgroup, and any two such maximal compact subgroups are conjugates of each other.*

Thus, we will often speak of "the" maximal compact subgroup, at least up to conjugation. For example, in the following table we list maximal compact subgroups of important matrix groups

$$\begin{aligned} GL_n(\mathbb{C}) &\rightsquigarrow U(n) \\ SL_n(\mathbb{C}) &\rightsquigarrow SU(n) \\ GL_n(\mathbb{R}) &\rightsquigarrow O_n(\mathbb{R}) \\ SL_n(\mathbb{R}) &\rightsquigarrow SO_n(\mathbb{R}) \end{aligned}$$

The maximal compact subgroup of $Sp_{2n}(\mathbb{R})$ is also isomorphic to $U(n)$, while the maximal compact subgroup of $Sp_{2n}(\mathbb{C})$ is its intersection with $U(2n)$ inside square matrices of size $2n$. Induction and restriction (which you learned in the context of finite groups in Math 314) allow one to relate the representations of a Lie group with those of its maximal compact subgroup. When it comes to Lie algebras, compact Lie groups are special for the following reason.

Theorem 2. *If G is a connected compact Lie group, then the exponential map*

$$\mathfrak{g} \to G$$

(that was defined in the exercise sessions) is surjective.

Proof. (*sketch*) An important and difficult result called the **Peter-Weyl theorem** implies that any compact Lie group G embeds into $U(n)$ for some $n > 0$. With this in mind, the exponential map $\mathfrak{g} \to G$ coincides with the restriction of the exponential map

$$\mathfrak{u}(n) \to U(n) \tag{45}$$

The map (45) is given by the usual matrix exponentiation, so it suffices to show that it is surjective. This is a well-known consequence of the fact that any unitary matrix is diagonalizable

$$g = P \cdot \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot P^{-1}$$

with P unitary and $\theta_1, \dots, \theta_n \in \mathbb{R}$. Therefore, the logarithm of g

$$x = P \cdot \mathrm{diag}(i\theta_1, \dots, i\theta_n) \cdot P^{-1}$$

is well-defined and lies in $\mathfrak{u}(n)$. □

3.4

Until now, we have presented results that were completely parallel between the real and complex settings. We will now show the rich interplay between these two settings.

Definition 13. *If $\mathfrak{k}$ is a real Lie algebra, we call*

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{k} \oplus \mathfrak{k}i$$

*(with the same Lie bracket, but extended to complex coefficients) the **complexification** of $\mathfrak{k}$.*

Conversely, if a complex Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{k}_{\mathbb{C}}$ for a real Lie algebra $\mathfrak{k}$, then we call $\mathfrak{k}$ a **real form** of $\mathfrak{g}$. Complexification is a powerful tool, but it is not always obvious, for instance

$$(\mathfrak{sl}_{n,\mathbb{R}})_{\mathbb{C}} \cong \mathfrak{sl}_{n,\mathbb{C}} \cong \mathfrak{su}(n)_{\mathbb{C}} \tag{46}$$

(in other words, $\mathfrak{sl}_{n,\mathbb{C}}$ has two interesting real forms). While the first isomorphism is obvious (just extend the coefficients of matrices from real to complex), we invite you to prove the second one.

Definition 14. *Let K be a compact real Lie group. A **complexification** of K is a complex Lie group G together with an embedding of smooth manifolds*

$$K \hookrightarrow G$$

which is universal with respect to real Lie group homomorphisms from K to complex Lie groups.

Above, universality means that any real Lie group homomorphism $K \to G'$ with G' a complex Lie group factors uniquely through a complex Lie group homomorphism $G \to G'$. The following result is very important, but its proof goes beyond the scope of our course.

Theorem 3. *Any compact real Lie group K admits a complexification G, whose maximal compact subgroup is K itself. The Lie algebra $\mathfrak{k} = \mathrm{Lie}(K)$ is a real form of $\mathfrak{g} = \mathrm{Lie}(G)$.*

As an example, recall that (46) stated that $\mathfrak{sl}_{n,\mathbb{C}}$ is a complexification of $\mathfrak{su}(n)$. Let us show that

$$SL_n(\mathbb{C}) \quad \text{is a complexification of} \quad SU(n) \tag{47}$$

Any compact subgroup of $SL_n(\mathbb{C})$ preserves an inner product on $\mathbb{C}^n$ by the argument in the proof of Proposition 5, so any such compact subgroup must be contained inside a conjugate of the unitary group; thus, we conclude that $SU(n)$ is a maximal compact subgroup of $SL_n(\mathbb{C})$. To see that (47) satisfies the universality property of Definition 14, a good way is to realize $SU(n)$ and $SL_n(\mathbb{C})$ as one and the same matrix group, but the former with real coefficients and the latter with complex coefficients. The solution is to consider

$$\left\{A, B \in \mathrm{Mat}_{n\times n} \middle| AA^T + BB^T = I_n, AB^T = BA^T, \det(A + Bi) = 1\right\} \tag{48}$$

with multiplication given by

$$(A, B)(A', B') = (AA' - BB', AB' + BA')$$

Check that the above multiplication defines a group. When A, B are real matrices, we see that $A + Bi$ is a unitary matrix, and so we recognize the above group as $SU(n)$. When A, B are complex matrices, one needs to check that any matrix $g \in SL_n(\mathbb{C})$ can be uniquely written as $A + Bi$, where A and B are complex matrices which satisfy the conditions in (48) (set $A = \frac{g+g^{T,-1}}{2}$, $B = \frac{g-g^{T,-1}}{2i}$).

Remark: as a partial converse to Theorem 3, we have the following (you are not expected to know what "semisimple" means yet, but you will learn in Lecture 7).

Theorem 4. *Suppose a complex Lie algebra $\mathfrak{g}$ is semisimple, in the sense of Definition 20. Then it has a real form $\mathfrak{k}$ which is the Lie algebra of a compact Lie group K. Moreover, if G is the connected complex Lie group with Lie algebra $\mathfrak{g}$, one can choose K to be a maximal compact subgroup of G.*

Lecture 4

4.1

In Lectures 1 and 2, we showed how to go from a Lie group (and its representations) to a Lie algebra (and its representations). We will now show how to go backward, and this will allow us to reduce the study of Lie groups to that of Lie algebras. This is beneficial because Lie algebras are "linear" objects, and thus very amenable to being studied using the tools of linear algebra.

Let G be a (real or complex) Lie group. A subgroup $H \subseteq G$ is called a

- Lie subgroup if $H \hookrightarrow G$ is an immersion
- closed Lie subgroup if $H \hookrightarrow G$ is an embedding

The terminology for closed Lie subgroups is motivated by the (non-obvious) fact that they are also closed as topological subspaces of G. Most Lie subgroups of interest will turn out to be closed, for example the stabilizers of Lie group actions $G \curvearrowright M$

$$\mathrm{Stab}_G(m) = \left\{ g \in G \middle| g \cdot m = m \right\}$$

are all closed Lie subgroups of G. However, there exist examples of non-closed Lie subgroups, e.g. the image of the group homomorphism $\mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2, t \mapsto (t, ct)$ for some $c \in \mathbb{R}\backslash\mathbb{Q}$. The following theorem is not a difficult result, but because it requires more differential geometry than we are willing to do in this course, its proof will be skipped.

Theorem 5. *(a) If $H \subseteq G$ is a normal closed Lie subgroup, then*

$$\boxed{G/H}$$

has an induced structure of a Lie group.

(b) If $f : G \to G'$ is a Lie group homomorphism, then Ker f *is a normal closed Lie subgroup, and we obtain an induced Lie group homomorphism*

$$G/\mathrm{Ker}\ f \hookrightarrow G'$$

which is an immersion. Im f *is a Lie subgroup of G' on general grounds; if it is moreover a closed Lie subgroup of G' then we have the following analogue of the first isomorphism theorem*

$$\boxed{G/\mathrm{Ker}\ f \cong \mathrm{Im}\ f}$$

(c) The center $Z(G)$ is a closed Lie subgroup.

4.2

Given a Lie algebra $\mathfrak{g}$ in the generality of Definition 9, a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a

- (Lie) **subalgebra** if it is closed under the Lie bracket, i.e. $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$;
- **ideal** if $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$. The kernel of any Lie algebra homomorphism is an ideal.

Check for yourself that if $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal, then $\mathfrak{g}/\mathfrak{h}$ inherits a Lie algebra structure. The following result is an analogue of the correspondence theorem that you saw for groups in Math 211.

Theorem 6. *If $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal, then there is a one-to-one correspondence*

$$\Big(\textit{Lie subalgebras } \mathfrak{h} \subseteq \mathfrak{a} \subseteq \mathfrak{g}\Big) \leftrightarrow \Big(\textit{Lie subalgebras } \bar{\mathfrak{a}} \subseteq \bar{\mathfrak{g}}\Big) \tag{49}$$

given by $\bar{\mathfrak{a}} = \pi(\mathfrak{a})$ and $\mathfrak{a} = \pi^{-1}(\bar{\mathfrak{a}})$, where

$$\pi : \mathfrak{g} \to \bar{\mathfrak{g}} := \mathfrak{g}/\mathfrak{h}$$

is the natural projection. In (49), $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ if and only if $\bar{\mathfrak{a}}$ is an ideal of $\bar{\mathfrak{g}}$.

One may also represent quotients in terms of **short exact sequences** of Lie algebras

$$0 \to \mathfrak{h} \xrightarrow{\phi} \mathfrak{g} \xrightarrow{\psi} \bar{\mathfrak{g}} \to 0 \tag{50}$$

with the implication being that Im ϕ is an ideal in $\mathfrak{g}$, and $\psi : \mathfrak{g}/\text{Im } \phi \to \bar{\mathfrak{g}}$ is an isomorphism.

Definition 15. *Given Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ over the same ground field, their* ***direct sum***

$$\boxed{\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2} \tag{51}$$

has Lie bracket defined by

$$\Big[(x_1, x_2), (y_1, y_2)\Big] = \Big([x_1, y_1], [x_2, y_2]\Big)$$

In other words, the subalgebras $\mathfrak{g}_1 \oplus 0$ and $0 \oplus \mathfrak{g}_2$ in (51) are ideals of $\mathfrak{g}$.

For example, show that we have an isomorphism of Lie algebras

$$\mathfrak{gl}_{n,\mathbb{K}} \cong \mathbb{K} \oplus \mathfrak{sl}_{n,\mathbb{K}} \tag{52}$$

over any ground field $\mathbb{K}$ of characteristic that does not divide n, where the right-hand side is the direct sum of the trivial one-dimensional Lie algebra (with Lie bracket zero) and $\mathfrak{sl}_{n,\mathbb{K}}$.

Definition 16. *Given an element x in a Lie algebra $\mathfrak{g}$, its* ***centralizer*** *is*

$$\boxed{\mathfrak{z}_x(\mathfrak{g}) = \Big\{y \in \mathfrak{g} \Big| [x, y] = 0\Big\}} \tag{53}$$

The intersection of all the centralizers is called the ***center*** *of $\mathfrak{g}$*

$$\boxed{\mathfrak{z}(\mathfrak{g}) = \Big\{y \in \mathfrak{g} \Big| [x, y] = 0, \ \forall x \in \mathfrak{g}\Big\}} \tag{54}$$

We include the following Theorem to complete the picture between subgroups and subalgebras; its proof follows the sketch of the proof of Theorem 8, so we skip it for now.

Theorem 7. *Let G be a real/complex Lie group with real/complex Lie algebra $\mathfrak{g}$.*

(a) If $H \subseteq G$ is a Lie subgroup, not necessarily closed, then $\mathfrak{h} = \mathrm{Lie}(H)$ is a Lie subalgebra of $\mathfrak{g}$ (the correspondence $H \rightsquigarrow \mathfrak{h}$ is invertible if we restrict to connected H).

(b) If $H \subseteq G$ is a normal closed Lie subgroup, then $\mathfrak{h} = \mathrm{Lie}(H)$ is an ideal of $\mathfrak{g}$, and

$$\boxed{\mathrm{Lie}(G/H) \cong \mathfrak{g}/\mathfrak{h}}$$

(c) $\mathrm{Lie}(Z(G)) = \mathfrak{z}(\mathfrak{g})$ if G is connected.

4.3

The above Theorem establishes a correspondence between subgroups of a Lie group G and subalgebras of $\mathfrak{g} = \mathrm{Lie}(G)$. The following result generalizes this fact.

Theorem 8. *(a) For any real/complex Lie groups G and G' (with G connected), there exists an injective assignment*

$$\Big\{\textit{Lie group homomorphisms } G \to G'\Big\} \rightsquigarrow \Big\{\textit{Lie algebra homomorphisms } \mathfrak{g} \to \mathfrak{g}'\Big\} \tag{55}$$

(given by derivative at the identity element) where $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}' = \mathrm{Lie}(G')$.

(b) If furthermore G is simply connected, then (55) is a bijection.

It is clear why Theorem 8 requires G to be connected, because $\mathfrak{g}$ only "knows" about the connected component of the identity in G. The simply-connected assumption is necessary to rule out examples like

$$S^1 = \left\{e^{2\pi i x} \middle| x \in \mathbb{R}\right\} \quad \text{with} \quad \mathrm{Lie}(S^1) = \mathbb{R}$$

(the operation on S^1 is multiplication, while the Lie bracket on $\mathbb{R}$ is zero), in which case

$$\mathrm{Hom}_{\text{Lie group}}(S^1, S^1) \cong \mathbb{Z} \quad \text{but} \quad \mathrm{Hom}_{\text{Lie algebra}}(\mathbb{R}, \mathbb{R}) = \mathbb{R}$$

Proof. of Theorem 8 (sketch): (a) The assignment (55) is given by taking the derivative, and we have already seen in Proposition 2 that it takes a Lie group homomorphism $f : G \to G'$ to the Lie algebra homomorphism $f_* : \mathfrak{g} \to \mathfrak{g}'$. By Exercise Sheet 2, Problem 3, we have

$$f(\exp(x)) = \exp(f_*(x))$$

for any x in a neighborhood of $0 \in \mathfrak{g}$. This means that knowledge of f_* determines f completely in a neighborhood of $e \in G$. However, any connected Lie group is generated by any neighborhood of the identity (it is not hard to show that the subgroup generated by any open subset must be open; if $H \subseteq G$ is the subset generated by an open neighborhood U of the identity, then $G \backslash H$ is also open, because for any $g \in G \backslash H$ we must have $gU \cap H = \emptyset$; since G is connected, this implies that $H = G$) so we conclude that f_* completely determines f, i.e. the assignment (55) is injective.

(b) Let us show that any Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ can be lifted to a Lie group homomorphism $f : G \to G'$, i.e. $\varphi = f_*$. To this end, we invoke the Baker-Campbell-Hausdorff formula from Exercise Sheet 2, Problem 5:

$$\exp(x)\exp(y) = \exp\left(x + y + \frac{[x,y]}{2} + \dots\right)$$

Then the function

$$f : G \to G', \qquad f(\exp(x)) = \exp(\varphi(x))$$

gives a Lie group homomorphism in a neighborhood of the identity $e \in U \subset G$, because

$$f(\exp(x)\exp(y)) = f\left(\exp\left(x + y + \frac{[x,y]}{2} + \dots\right)\right) = \exp\left(\varphi(x) + \varphi(y) + \varphi\left(\frac{[x,y]}{2}\right) + \dots\right) =$$

$$= \exp\left(\varphi(x) + \varphi(y) + \frac{[\varphi(x),\varphi(y)]}{2} + \dots\right) = \exp(\varphi(x))\exp(\varphi(y)) = f(\exp(x))f(\exp(y))$$

As we have seen in part (a), the group G is generated by U. Thus, for any $g \in G$ we can choose

$$e = g_0, g_1, \dots, g_{k-1}, g_k = g \tag{56}$$

where each g_{i-1} is close enough to g_i so that $g_i g_{i-1}^{-1} \in U$. This means that we can define

$$f(g) = f(g_k g_{k-1}^{-1}) f(g_{k-1} g_{k-2}^{-1}) \dots f(g_2 g_1^{-1}) f(g_1 g_0^{-1}) \tag{57}$$

To show that this is well-defined, the key observation is that the value of $f(g)$ above is independent of the choice of (56). To see this, consider any

$$e = g_0, g_1, \dots, g_{k-1}, g_k = g = g'_{k'}, g'_{k'-1}, \dots, g'_1, g'_0 = e$$

then string a path through the g_i's and a path through the g'_i's, and then take a homotopy between the two paths (which exists because G is simply connected). We leave the details to you. The independence of (57) on the choice of (56) also proves $f(gh) = f(g)f(h)$, because one can construct a sequence (56) from e to gh by stringing together an analogous sequence for g and one for h. □

4.4

The following result is sometimes known as Lie's third theorem.

Theorem 9. *Any finite-dimensional real/complex Lie algebra* $\mathfrak{g}$ *has the property that*

$$\mathfrak{g} \cong \mathrm{Lie}(G)$$

for a unique (up to isomorphism) simply connected real/complex Lie group G*.*

Proof. (sketch): This is a quite difficult result, unless one accepts **Ado's theorem**: any finite-dimensional Lie algebra (over any field) has a **faithful** finite-dimensional representation, i.e. one such that (27) is injective. Therefore, we may assume that our Lie algebra satisfies

$$\mathfrak{g} \subseteq \mathfrak{gl}_{n,\mathbb{K}}$$

for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $G' \subset GL_n(\mathbb{K})$ to be the closure of the subgroup generated by $\{\exp(X)\}_{X \in \mathfrak{g}}$ with respect to the usual matrix exponential. To obtain a simply connected Lie group, we define

$$G = \Big\{ \text{paths } \gamma : [0,1] \to G, \gamma(0) = e \Big\} \Big/ \text{homotopy}$$

made into a Lie group with respect to pointwise multiplication of paths. The covering map

$$G \to G', \qquad \gamma \mapsto \gamma(1)$$

is a Lie group homomorphism. It is well-known that G and G' have the same tangent space at the identity, which is $\mathfrak{g}$ by construction. The uniqueness is a special case of Theorem 8. ☐

4.5

A particularly important special case of the results in the previous Subsections arises in the study of Lie group representations

$$G \curvearrowright V \quad \Leftrightarrow \quad \text{Lie group homomorphisms } G \to GL(V) \tag{58}$$

Theorem 8 implies that if G is connected (for example $SL_n(\mathbb{R})$, $SO_n(\mathbb{R})$, $Sp_{2n}(\mathbb{R})$, $U(n)$, $SU(n)$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$), then such a representation is completely determined by the induced representation of $\mathfrak{g} = \text{Lie}(G)$

$$\mathfrak{g} \curvearrowright V \quad \Leftrightarrow \quad \text{Lie algebra homomorphisms } \mathfrak{g} \to \mathfrak{gl}(V) \tag{59}$$

If moreover G is simply connected (for example $SU(n), SL_n(\mathbb{C}), Sp_{2n}(\mathbb{C})$), then any representation (59) can be uniquely lifted to a representation (58). This is very convenient, as it reduces the study of Lie group representations (which is a more complicated problem that interweaves algebra and geometry) to the purely linear algebraic problem of studying Lie algebra representations. Therefore, in the remainder of this course we will only study Lie algebras and their representations.

Lecture 5

5.1

We henceforth only consider complex Lie algebras, and thus drop the subscript $\mathbb{C}$ from our notation. Before we develop the general theory of complex Lie algebras, let us focus on the simplest example

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{C}, a + d = 0 \right\} \tag{60}$$

and study its representations (all representations will henceforth be assumed complex). The machinery developed in the previous Lectures allows us to prove that any finite-dimensional representation $\mathfrak{sl}_2 \curvearrowright V$ is completely reducible, i.e.

$$V \cong V_1 \oplus \dots \oplus V_k \tag{61}$$

where V_i are all irreducible representations. The argument goes as follows: V restricts to a representation of the real form $\mathfrak{su}(2)$ of $\mathfrak{sl}_2$, and so V can be promoted to a Lie group representation of $SU(2)$ by the discussion in Subsection 4.5. Since $SU(2)$ is compact, Proposition 5 and Subsection 3.1 imply a decomposition (61) as representations of $SU(2)$. Using the discussion in Subsection 4.5 again allows us to consider the decomposition in question as one of representations of $\mathfrak{su}(2)$, and Exercise ... allows us to lift this decomposition to one of representations of $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}_2$.

5.2

We will now give a purely algebraic (and very explicit) description of the finite-dimensional representations of $\mathfrak{sl}_2$. We begin by observing that $\mathfrak{sl}_2$ is a three-dimensional complex Lie algebra

$$\mathfrak{sl}_2 = \mathbb{C}E \oplus \mathbb{C}H \oplus \mathbb{C}F$$

where

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{62}$$

The Lie bracket can be easily computed from commutators of matrices

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H \tag{63}$$

Any Lie algebra has the **trivial representation** $\mathbb{C}$ on which all elements act by 0, and we will denote the trivial representation of $\mathfrak{sl}_2$ by $L(0)$. Moreover, we can construct a two-dimensional representation $\mathfrak{sl}_2 \curvearrowright L(1) = \mathbb{C}^2$ by considering the natural action of 2×2 matrices (60) on $\mathbb{C}^2$. Convince yourselves that $L(0)$ and $L(1)$ are both irreducible representations.

5.3

For any $n \geq 0$, the action $\mathfrak{sl}_2 \curvearrowright \mathbb{C}^2$ generalizes to the n-th symmetric power of $\mathbb{C}^2$

$$S^n\mathbb{C}^2 = \left\{ \text{linear combinations of } w_1 \otimes \dots \otimes w_n \middle| w_1, \dots, w_n \in \mathbb{C}^2 \right\} \Big/ \Big(\cdots w \otimes w' \cdots - \cdots w' \otimes w \cdots \Big)$$

If we let e_1, e_2 be the standard basis of $\mathbb{C}^2$, then we obtain a basis of the n-th symmetric power

$$S^n\mathbb{C}^2 = \bigoplus_{i=0}^{n} \mathbb{C}v_i, \qquad \text{where} \qquad v_i = e_1^{\otimes i} \otimes e_2^{\otimes n-i}$$

and so $S^n\mathbb{C}^2$ has dimension $n+1$. Moreover, the action $\mathfrak{sl}_2 \curvearrowright \mathbb{C}^2$ extends to an action

$$\boxed{\mathfrak{sl}_2 \curvearrowright L(n) = S^n\mathbb{C}^2}$$

given by the formula $x \cdot (w_1 \otimes \cdots \otimes w_n) = \sum_{i=1}^n w_1 \otimes \cdots \otimes x(w_i) \otimes \cdots \otimes w_n$ and explicitly by

$$Ev_i = (n-i)v_{i+1} \tag{64}$$

$$Fv_i = iv_{i-1} \tag{65}$$

$$Hv_i = (2i-n)v_i \tag{66}$$

It is easy to check from the formulas above that $L(n)$ is irreducible for all n, and that H acts by a diagonalizable operator. We will soon see that the latter is actually a general feature. Recall that unless otherwise stated, all our representations will be complex.

Proposition 6. *Any $n+1$ dimensional irreducible representation of $\mathfrak{sl}_2$ is isomorphic to $L(n)$.*

Proof. Consider an irreducible representation $\mathfrak{sl}_2 \curvearrowright V$ and let us consider the eigenspaces of H

$$V_\ell = \left\{ v \in V \middle| H \cdot v = \ell v \right\}$$

It is an easy consequence of (63) that

$$E \cdot V_\ell \subseteq V_{\ell+2}$$

$$F \cdot V_\ell \subseteq V_{\ell-2}$$

for all ℓ. This implies that the direct sum of all V_ℓ's is a subrepresentation of V, which must be non-zero because H has at least one non-zero eigenvector (this is where we need to use the fact that our representations are complex). Because V is irreducible, we conclude that

$$V = \bigoplus_{\ell \in \mathbb{C}} V_\ell$$

Because V is finite-dimensional, only finitely many of the V_ℓ's are nonzero. Let us consider a maximal such ℓ, i.e. such that $V_\ell \neq 0$ but $V_{\ell+2} = 0$. For any $0 \neq v \in V_\ell$, show using (63) that

$$v, Fv, F^2v, \ldots, F^nv \tag{67}$$

form a subrepresentation of V, where $n+1$ is the smallest positive integer such that $F^{n+1}v = 0$. The vectors (67) are linearly independent, because they lie in different eigenspaces of H. Because V is irreducible, we conclude that

$$V = \bigoplus_{i=0}^{n} \mathbb{C}F^i v$$

It is easy to check that the assignment $F^i v \mapsto \frac{v_{n-i}}{(n-i)!}$ gives an isomorphism $V \cong L(n)$.

$\square$

5.4

Having fully characterized the irreducible representations of $\mathfrak{sl}_2$, let us now characterize general finite-dimensional representations. As explained in Subsection 5.1, any finite-dimensional representation $\mathfrak{sl}_2 \curvearrowright V$ splits up as a direct sum of irreducible representations. By Proposition 6, there exist $n_1, \ldots, n_k \in \mathbb{N}$ such that

$$V \cong L(n_1) \oplus \cdots \oplus L(n_k) \tag{68}$$

This property is called the **complete reducibility** of finite-dimensional representations of $\mathfrak{sl}_2$.

Proposition 7. *Any finite-dimensional representation* $\mathfrak{sl}_2 \curvearrowright V$ *has a decomposition*

$$V = \bigoplus_{\ell \in \mathbb{Z}} V_\ell \tag{69}$$

where $V_\ell = \{v \in V | Hv = \ell v\}$. *Those* ℓ*'s such that* $V_\ell \neq 0$ *in the above formula are called the* ***weights*** *of* V*, and the corresponding* V_ℓ *are called the* ***weight spaces****.*

Note how different the situation would have been if we replaced $\mathfrak{sl}_2$ by the one-dimensional Lie algebra $\mathbb{C}H \cong \mathfrak{gl}_1$. Since a representation of the latter Lie algebra boils down to an abritrary linear transformation H on a vector space, there would be no restriction on the eigenvalues of such an H. In stark contrast, the presence of E and F in the Lie algebra $\mathfrak{sl}_2$ forces H to act as a diagonalizable matrix with integer eigenvalues.

Proof. of Proposition 7: By (68), it suffices to prove the result for $V = L(n)$. In this case, we saw in (66) that the irreducible representation $L(n)$ has a weight decomposition (69) with weights

$$n, n-2, \ldots, 2-n, -n \tag{70}$$

□

Because the weights of irreducible representations are symmetric around the origin, and because every finite-dimensional representations of $\mathfrak{sl}_2$ is a direct sum of the form (68), we have the following consequence (which plays an important part in the hard Lefschetz theorem in algebraic geometry).

Corollary 1. *If* V *is a finite-dimensional representation of* $\mathfrak{sl}_2$*, then for all* $k \in \mathbb{N}$*, the linear transformations* E^k *and* F^k *give isomorphisms between the* k*-th and* $-k$*-th weight subspaces of* V*.*

Corollary 2. *Assume that a representation* $\mathfrak{sl}_2 \curvearrowright V$ *has finite-dimensional weight subspaces, and the subrepresentation generated by every vector is finite-dimensional. Then* V *is finite-dimensional.*

Proof. If V were not finite-dimensional, then we can inductively construct a subrepresentation

$$L(n_1) \oplus \cdots \oplus L(n_k) \hookrightarrow V \tag{71}$$

for arbitrarily large $k \in \mathbb{N}$. Indeed, consider a subrepresentation as in (71) and let v be a vector not inside it. By the hypothesis, v generates a finite-dimensional $\mathfrak{sl}_2$ representation W. Letting $\bar{W}$

be the sum of W and $L(n_1) \oplus \dots \oplus L(n_k)$ means that $\bar{W}$ is a finite-dimensional representation of $\mathfrak{sl}_2$, hence completely reducible. We may then find natural numbers $n_{k+1}, \dots, n_{k+k'}$ such that

$$L(n_1) \oplus \dots \oplus L(n_{k+k'}) \cong \bar{W} \hookrightarrow V$$

This concludes the proof of the fact that there exists an inclusion (71) for arbitrarily large k. However, every one of the irreducible representations in (71) contributes a dimension of one to either the 0-th or the 1-st weight subspace of V, by (70). Since the latter are finite-dimensional by assumption, we obtain a contradiction. $\square$

5.5

We will now define the **Casimir operator** for $\mathfrak{sl}_2$, which will be introduced in all its glory in Subsection 8.4. Consider any representation

$$\mathfrak{sl}_2 \curvearrowright V$$

and consider the linear transformation

$$C = EF + FE + \frac{H^2}{2} : V \to V \tag{72}$$

(above, we slightly abuse notation by writing E, F, H for the linear transformations on V induced by the same-named elements of $\mathfrak{sl}_2$). We stress the fact that C is not an element of $\mathfrak{sl}_2$, but it can be understood as an element in the universal enveloping algebra of $\mathfrak{sl}_2$, as in the following Lecture. In the meantime, let us observe that the defining relations in the Lie algebra $\mathfrak{sl}_2$ imply that

$$C = 2FE + H + \frac{H^2}{2} \tag{73}$$

More importantly, we have the following property.

Proposition 8. *The operator C commutes with E, F, H. In particular, if V is irreducible, then C is a scalar multiple of the identity (by Schur's Lemma).*

Proof. The Proposition is easy, but very important, so we present the details. Formulas (63) and the Leibniz rule for commutators of products imply that

$$[E, C] = E[E, F] + [E, F]E + \frac{H[E, H]}{2} + \frac{[E, H]H}{2} = EH + HE - HE - EH = 0$$
$$[F, C] = [F, E]F + F[F, E] + \frac{H[F, H]}{2} + \frac{[F, H]H}{2} = -HF - FH + HF + FH = 0$$
$$[H, C] = [H, E]F + E[H, F] + [H, F]E + F[H, E] = 2EF - 2EF - 2FE + 2FE = 0$$

$\square$

To compute the constant by which C acts on the irreducible representation $L(n)$, it suffices to compute its action on the highest weight vector v_n. Because $Ev_n = 0$, then (73) implies that

$$Cv_n = \frac{n(n+2)}{2} v_n$$

so we conclude that the constant in question is $\frac{n(n+2)}{2}$. For later purposes, it will be important to calculate the action of the terms EF and FE on the ℓ-th weight subspace of $L(n)$ for all $\ell \in \mathbb{Z}$, which is also quite easy from formulas (64)-(65)

$$EFv_i = i(n-i+1)v_i \quad \Rightarrow \quad EF\Big|_{\text{weight } \ell} = \frac{(n+\ell)(n-\ell+2)}{4} \tag{74}$$

$$FEv_i = (i+1)(n-i)v_i \quad \Rightarrow \quad FE\Big|_{\text{weight } \ell} = \frac{(n+\ell+2)(n-\ell)}{4} \tag{75}$$

$$\frac{H^2}{2}v_i = \frac{(2i-n)^2}{2}v_i \quad \Rightarrow \quad \frac{H^2}{2}\Big|_{\text{weight } \ell} = \frac{\ell^2}{2} \tag{76}$$

This gives us another proof of $C|_{\text{weight } \ell} = \frac{(n+\ell)(n-\ell+2)}{4} + \frac{(n+\ell+2)(n-\ell)}{4} + \frac{\ell^2}{2} = \frac{n(n+2)}{2}$, $\forall \ell$.

5.6

Let us finally present the character theory of $\mathfrak{sl}_2$, which will give us a very elegant way to determine the numbers $n_1, \dots, n_k$ in (68). Specifically, for a finite-dimensional representation with weight decomposition (69), we define its **character** as the formal series

$$\chi_V = \sum_{\ell \in \mathbb{Z}} (\dim V_\ell) z^\ell$$

It is easy to check that the character satisfies the properties

$$\chi_{V \oplus V'} = \chi_V + \chi_{V'} \tag{77}$$

$$\chi_{V \otimes V'} = \chi_V \chi_{V'} \tag{78}$$

$$\chi_{V^\vee} = \overline{\chi_V} \tag{79}$$

(where $\overline{z^\ell} = z^{-\ell}$) with respect to direct sum, tensor product and dual representations, see (37), (38), (39). Moreover, the explicit description of the weight spaces of $L(n)$ in (70) shows that

$$\chi_{L(n)} = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} \tag{80}$$

By (77), the character of an arbitrary finite-dimensional representation V that decomposes as (68) would be

$$\chi_V = \sum_{i=1}^{k} \frac{z^{n_i+1} - z^{-n_i-1}}{z - z^{-1}}$$

and it is easy to see that one can extract the set of non-negative integers $\{n_1, \dots, n_k\}$ from χ_V. Thus, the decomposition (68) is unique up to reordering the summands, and is completely determined by χ_V. This allows us to prove the following formula for all $m \geq n$, known as the **Clebsch-Gordan** rule

$$L(m) \otimes L(n) \cong L(m+n) \oplus L(m+n-2) \oplus \cdots \oplus L(m-n+2) \oplus L(m-n) \tag{81}$$

simply by showing that the left and right-hand sides have the same character (use (77) and (78)).

Lecture 6

6.1

We will now systematically study Lie algebras $\mathfrak{g}$ and their representations

$$\mathfrak{g} \curvearrowright V \tag{82}$$

First of all, while the notion of Lie algebra representations may seem strange at first (for any $x \in \mathfrak{g}$ you get a linear transformation $\phi_x : V \to V$ such that

$$\phi_{[x,y]} = \phi_x \circ \phi_y - \phi_y \circ \phi_x$$

for all $x, y \in \mathfrak{g}$), it is actually a particular case of the familiar notion of an algebra representation.

Definition 17. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$. Make the vector space*

$$T\mathfrak{g} = \mathbb{K} \bigoplus \mathfrak{g} \bigoplus \mathfrak{g} \otimes \mathfrak{g} \bigoplus \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \bigoplus \dots$$

into an algebra (i.e. a ring with $\mathbb{K}$ inside) via concatenation of tensors. Then the quotient

$$\boxed{U\mathfrak{g} = T\mathfrak{g} \Big/ \Big(x \otimes y - y \otimes x - [x,y]\Big)_{x,y \in \mathfrak{g}}} \tag{83}$$

is called the ***universal enveloping algebra*** *of $\mathfrak{g}$.*

Note that in the right-hand side of (83), we factor by the two-sided ideal consisting of linear combinations of tensors

$$t_1 \otimes x \otimes y \otimes t_2 - t_1 \otimes y \otimes x \otimes t_2 - t_1 \otimes [x,y] \otimes t_2 \tag{84}$$

for any tensors $t_1, t_2 \in T\mathfrak{g}$ and any $x, y \in \mathfrak{g}$. The effect this has on $U\mathfrak{g}$ is to ensure that $[x,y]$ is identified with the commutator $x \otimes y - y \otimes x$ in all formulas. While the construction of the universal enveloping algebra might seem a little dry, prove by yourself the fact that

$$\mathfrak{g} \curvearrowright V \text{ is a Lie algebra representation} \qquad \Leftrightarrow \qquad U\mathfrak{g} \curvearrowright V \text{ is an algebra module}$$

In the right-hand side, recall that V being a module of the algebra $U\mathfrak{g}$ means that we have for all $z \in U\mathfrak{g}$ a linear transformation $\phi_z : V \to V$ such that

$$\phi_{zw} = \phi_z \circ \phi_w \quad \text{and} \quad \phi_1 = \mathrm{Id}_V$$

6.2

In general, quotients of non-commutative algebras such as (83) are quite badly behaved, e.g. they could have zero divisors. However, this is not true for universal enveloping algebras of Lie algebras due to an important structural result known as the Poincaré-Birkhoff-Witt (PBW) theorem. In a nutshell, this theorem starts from a $\mathbb{K}$-basis

$$x_1, \dots, x_n \tag{85}$$

of $\mathfrak{g}$ as a vector space, and claims that the symbols

$$x_1^{a_1} \dots x_n^{a_n} = \underbrace{x_1 \otimes \dots \otimes x_1}_{a_1 \text{ factors}} \otimes \underbrace{x_2 \otimes \dots \otimes x_2}_{a_2 \text{ factors}} \otimes \dots \otimes \underbrace{x_n \otimes \dots \otimes x_n}_{a_n \text{ factors}} \tag{86}$$

give rise to a basis of $U\mathfrak{g}$, as $a_1, \dots, a_n$ range over the non-negative integers.

Theorem 10. *We have a vector space isomorphism*

$$U\mathfrak{g} = \bigoplus_{a_1,\dots,a_n=0}^{\infty} \mathbb{K} \cdot x_1^{a_1} \dots x_n^{a_n} \tag{87}$$

Let us show that the symbols $x_1^{a_1} \dots x_n^{a_n}$ span $U\mathfrak{g}$ (the fact that they are linearly independent is more difficult, and we will not prove it). By definition, $U\mathfrak{g}$ is spanned by tensors of the form $x_{i_1} \otimes \dots \otimes x_{i_k}$ where $1 \leq i_1, \dots, i_k \leq n$. We are trying to prove the fact that any such tensor can be "ordered" so as to have $i_1 \leq \dots \leq i_k$. If at some point we have $i_s > i_{s+1}$, we apply the equality

$$\Big(\dots \otimes x_{i_s} \otimes x_{i_{s+1}} \otimes \dots\Big) = \Big(\dots \otimes x_{i_{s+1}} \otimes x_{i_s} \otimes \dots\Big) + \Big(\dots \otimes [x_{i_s}, x_{i_{s+1}}] \otimes \dots\Big)$$

One can re-express the Lie bracket $[x_{i_s}, x_{i_{s+1}}]$ as a linear combination of x_j's, and note that the right-most term in the above expression has $k-1$ tensor factors. Thus, after finitely many such steps, any tensor $x_{i_1} \otimes \dots \otimes x_{i_k}$ can be written as a linear combination of ordered tensors.

Note that the symbols (86) run over a basis of the symmetric algebra

$$S\mathfrak{g} = \mathbb{K} \bigoplus \mathfrak{g} \bigoplus S^2\mathfrak{g} \bigoplus S^3\mathfrak{g} \bigoplus \dots$$

Therefore, (87) is an isomorphism of vector spaces

$$U\mathfrak{g} \cong S\mathfrak{g} \tag{88}$$

However, we note that (88) is not an isomorphism of graded vector spaces. Indeed, while $S\mathfrak{g}$ and $T\mathfrak{g}$ are graded algebras (with $S^n\mathfrak{g}$ and $\mathfrak{g}^{\otimes n}$ in degree n), the quotient $U\mathfrak{g}$ is not graded because we set the degree 2 element $x \otimes y - y \otimes x$ equal to the degree 1 element $[x, y]$. However, $U\mathfrak{g}$ is a **filtered algebra**, i.e. there exists a sequence of subspaces

$$U_0\mathfrak{g} \subset U_1\mathfrak{g} \subset \dots \subset U_n\mathfrak{g} \subset \dots \subset U\mathfrak{g} \quad \text{such that } U\mathfrak{g} = \bigcup_{n=0}^{\infty} U_n\mathfrak{g}$$

such that $U_k\mathfrak{g} \cdot U_\ell\mathfrak{g} \subseteq U_{k+\ell}\mathfrak{g}$. Indeed, the natural choice is to define $U_k\mathfrak{g}$ as the image of $\oplus_{i=0}^{k}\mathfrak{g}^{\otimes i}$ in the quotient (83), which is a good idea because any commutation relation between $\leq k$ tensor factors will also involve $\leq k$ tensor factors.

Proposition 9. *For any $x \in U_k\mathfrak{g}$ and $y \in U_\ell\mathfrak{g}$, we have*

$$xy - yx \in U_{k+\ell-1}\mathfrak{g}$$

Therefore, the induced associated graded algebra

$$\text{gr } U\mathfrak{g} = \bigoplus_{n=0}^{\infty} \Big(U_n\mathfrak{g} \Big/ U_{n-1}\mathfrak{g}\Big)$$

is commutative.

Proof. Assume $x = x_1 \otimes \dots \otimes x_k$ and $y = y_1 \otimes \dots \otimes y_\ell$. Then the commutator $xy - yx$ equals

$$x_1 \otimes \dots \otimes x_k \otimes y_1 \otimes \dots \otimes y_\ell - y_1 \otimes \dots \otimes y_\ell \otimes x_1 \otimes \dots \otimes x_k =$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{\ell} x_1 \otimes \dots \otimes x_{i-1} \otimes y_1 \otimes \dots \otimes y_{j-1} \otimes \underbrace{\Big(x_i \otimes y_j - y_j \otimes x_i \Big)}_{=[x_i, y_j]} \otimes y_{j+1} \otimes \dots \otimes y_\ell \otimes x_{i+1} \otimes \dots \otimes x_k$$

which is clearly in $U_{k+\ell-1}\mathfrak{g}$.

□

With Proposition 9 in mind, we can upgrade the Poincaré-Birkhoff-Witt theorem to the existence of an isomorphism of graded algebras

$$\boxed{\text{gr } U\mathfrak{g} \cong S\mathfrak{g}} \tag{89}$$

which sends $x_{i_1} \otimes \dots \otimes x_{i_k}$ to $x_{i_1} \dots x_{i_k}$. In particular, this shows that the natural composition

$$\mathfrak{g} \hookrightarrow T\mathfrak{g} \twoheadrightarrow U\mathfrak{g} \tag{90}$$

is injective, which is not obvious from the mere fact that $U\mathfrak{g}$ is a quotient of $T\mathfrak{g}$ by an ideal.

Example 4. *When $\mathfrak{g} = \mathfrak{sl}_2$, the universal enveloping algebra is quite simple*

$$U\mathfrak{sl}_2 = \mathbb{K}\Big\langle E, F, H \Big\rangle \Big/ \left(HE = E(H+2), HF = F(H-2), EF - FE = H\right)$$

This allows us to construct numerous representations of $\mathfrak{sl}_2$, such as for any $\lambda \in \mathbb{K}$

$$M(\lambda) = \mathbb{K}[F] = \mathbb{K}1 \oplus \mathbb{K}F \oplus \mathbb{K}F^2 \oplus \dots \tag{91}$$

via $E1 = 0$, $H1 = \lambda$ and F acting by multiplication. The rest of the action is determined by the defining relations of $U\mathfrak{sl}_2$. The above infinite-dimensional representation has weights $\lambda, \lambda - 2, \lambda - 4, \dots,$ in stark contrast with finite-dimensional representations, which we have seen have weights which are all integers and symmetric around 0. In Lecture 13, we will see that (91) *is an example of a Verma module.*

6.3

We will now introduce the natural Lie algebra versions (over any ground field $\mathbb{K}$) of the abelian, solvable and nilpotent groups that you studied in Math 211 . We start by calling a Lie algebra $\mathfrak{g}$ **abelian** if its Lie bracket is identically 0, which is equivalent to $U\mathfrak{g}$ being commutative.

Definition 18. *A Lie algebra $\mathfrak{g}$ is called **solvable** if it has a chain of Lie subalgebras*

$$0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \dots \subset \mathfrak{g}_k = \mathfrak{g}$$

such that $\mathfrak{g}_{i-1}$ is an ideal in $\mathfrak{g}_i$ and $\mathfrak{g}_i / \mathfrak{g}_{i-1}$ is abelian for all $i \in \{1, \dots, k\}$.

As in group theory, Definition 18 may be re-expressed in terms of commutators. Given subspaces A, B of a Lie algebra $\mathfrak{g}$, we will write

$$[A, B] = \text{span}\left\{[a, b]\middle|a \in A, b \in B\right\}$$

In particular, the **derived Lie subalgebra** of $\mathfrak{g}$ is

$$\boxed{D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]} \tag{92}$$

Prove that $D\mathfrak{g}$ is an ideal. Moreover, it is a well-known fact, which we invite you to prove, that $\mathfrak{g}$ is solvable if and only if the so-called **derived** series

$$\mathfrak{g} \supseteq D\mathfrak{g} \supseteq D(D\mathfrak{g}) \supseteq \dots \tag{93}$$

eventually terminates with the 0 subalgebra.

Definition 19. *A Lie algebra $\mathfrak{g}$ is called **nilpotent** if it has a chain of ideals*

$$0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \dots \subset \mathfrak{g}_k = \mathfrak{g}$$

such that $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i-1}$ for all $i \in \{1, \dots, k\}$.

It is a well-known fact, which we invite you to prove, that $\mathfrak{g}$ is nilpotent if and only if the series

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supseteq \dots \tag{94}$$

(called the **lower central series**) eventually terminates with the 0 subalgebra. Since a Lie algebra is abelian if and only if $[\mathfrak{g}, \mathfrak{g}] = 0$, we conclude that

$$\text{abelian} \subset \text{nilpotent} \subset \text{solvable}$$

holds in the world of Lie algebras as it did in the case of groups. The fundamental example of abelian, nilpotent and solvable Lie algebras are

$$\mathfrak{h} = \left\{\text{diagonal matrices}\right\} \tag{95}$$

$$\mathfrak{n} = \left\{\text{strictly upper triangular matrices}\right\} \tag{96}$$

$$\mathfrak{b} = \left\{\text{upper triangular matrices}\right\} \tag{97}$$

respectively, all regarded as Lie subalgebras of $\mathfrak{gl}_n$ with the usual Lie bracket commutator. It is a good idea to check the above statements. The following result is proved just like in group theory.

Proposition 10. *Any subalgebra or quotient of an abelian / nilpotent / solvable Lie algebra is abelian / nilpotent / solvable. Conversely, if we have an ideal*

$$\mathfrak{i} \subseteq \mathfrak{g}$$

such that $\mathfrak{i}$ and $\mathfrak{g}/\mathfrak{i}$ are solvable, then $\mathfrak{g}$ is solvable (in order to have this property for nilpotent Lie algebra, we would need to further assume that $\mathfrak{i}$ lies in the center of $\mathfrak{g}$).

6.4

For the remainder of this Lecture, we assume that the ground field $\mathbb{K}$ is algebraically closed. Under this assumption, it is well-known that a commutative family of endomorphisms of a finite-dimensional vector space can be simultaneously triangularized. In particular, if an abelian Lie algebra $\mathfrak{g}$ acts on a representation V, there exists a basis of V so that the action is given by upper triangular matrices. It turns out that the same is true for solvable Lie algebras, as in the following important result called **Lie's theorem**.

Theorem 11. *If a solvable Lie algebra $\mathfrak{g}$ acts on a finite-dimensional representation V (over an algebraically closed field), then there exists a basis of V so that the action is given by upper triangular matrices.*

Proof. Let us write $\{\phi_x\}_{x\in\mathfrak{g}} \in \mathrm{End}(V)$ for the operators that encode the action $\mathfrak{g} \curvearrowright V$. It suffices to show that all the ϕ_x have a joint eigenvector $v \in V$, because then one can obtain the desired result by induction on $\dim V$ (we can obtain a full flag of subspaces of V which is preserved by the operators ϕ_x by taking a full flag of subspaces of $V/\mathbb{K}v$ and appending $\mathbb{K}v$ to it). We will prove the aforementioned claim by induction on $\dim \mathfrak{g}$. Since $\mathfrak{g}$ is solvable, we have

$$D\mathfrak{g} \subsetneq \mathfrak{g}$$

so the quotient $\mathfrak{g}/D\mathfrak{g}$ is a non-zero abelian Lie algebra. Therefore, any codimension 1 subspace in $\mathfrak{g}/D\mathfrak{g}$ is an ideal, and by the correspondence theorem we conclude that there exists a codimension 1 ideal $\mathfrak{i} \subset \mathfrak{g}$. By Proposition 10, $\mathfrak{i}$ is a solvable Lie algebra, and so the induction hypothesis implies that there exists $0 \neq v \in V$ such that

$$x \cdot v = \lambda(x)v \tag{98}$$

for all $x \in \mathfrak{i}$, where λ is a linear functional on $\mathfrak{i}$. Pick $y \in \mathfrak{g}\backslash\mathfrak{i}$ and consider the subspace

$$W = \mathrm{span}\Big\{v, y \cdot v, y^2 \cdot v, \dots\Big\}$$

For any $x \in \mathfrak{i}$ and $k \geq 0$, you can prove the following elementary identity in $U\mathfrak{g}$

$$xy^k = y^k x + \sum_{i=1}^{k} \binom{k}{i} y^{k-i} \underbrace{[\dots[[x,y],y],\dots,y]}_{i\ y\text{'s}} \tag{99}$$

Since all the commutators in the right-hand side lie in $\mathfrak{i}$, applying the above relation to v yields

$$xy^k \cdot v = \lambda(x)y^k \cdot v + \sum_{i=1}^{k} \binom{k}{i} \lambda(\underbrace{[\dots[[x,y],y],\dots,y]}_{i\ y\text{'s}}) y^{k-i} \cdot v$$

Thus, any $x \in \mathfrak{i}$ acts on W upper triangularly in the basis $v, y \cdot v, \dots, y^{\dim W - 1} \cdot v$ of W, with the constant $\lambda(x)$ on the diagonal. Therefore,

$$\mathrm{tr}\,(x|_W) = \lambda(x)\dim W$$

for all $x \in \mathfrak{i}$. However, y also preserves W, and $[\mathfrak{i}, y] \subseteq \mathfrak{i}$ on account of $\mathfrak{i}$ being an ideal of $\mathfrak{g}$. The fact that commutators have trace 0 implies that

$$\lambda([x,y]) = 0 \tag{100}$$

for all $x \in \mathfrak{i}$. Let us now consider the non-zero subspace

$$W' = \left\{ v \in V \middle| x \cdot v = \lambda(x) v, \ \forall x \in \mathfrak{i} \right\}$$

for the same linear functional as in (98). For any $x \in \mathfrak{i}$ and $w \in W'$, we note that

$$x \cdot (y \cdot w) = y \cdot (x \cdot w) + [x, y] \cdot w = \lambda(x) y \cdot w + \lambda([x, y]) w$$

Since the second term in the right-hand side vanishes by (100), we conclude that $yw \in W'$. Thus, the action of y preserves W', so it has an eigenvector $w' \in W'$. This w' will be an eigenvector for the whole of $\mathfrak{g} = \mathfrak{i} \oplus \mathbb{K}y$. □

Corollary 3. *All irreducible finite-dimensional representations (over an algebraically closed field) of a solvable Lie algebra are one-dimensional.*

Lecture 7

7.1

In the last lecture, we saw that if V is a representation of a solvable Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbb{K}$, then the action Lie algebra homomorphism

$$\mathfrak{g} \to \mathfrak{gl}(V)$$

lands in the Lie subalgebra $\mathfrak{b} \subset \mathfrak{gl}(V)$ of upper triangular matrices (with respect to some basis). The analogous result for nilpotent Lie algebras and strictly upper triangular matrices is false (for example, V being a one-dimensional representation of a one-dimensional Lie algebra), but we have the following replacement. In what follows, we assume char $\mathbb{K} = 0$, but $\mathbb{K}$ needn't be algebraically closed.

Theorem 12. *If a Lie algebra $\mathfrak{g}$ acts on a finite-dimensional representation V by nilpotent operators, then there exists a basis of V so that the action is given by strictly upper triangular matrices.*

Proof. It suffices to show that all the operators $\{\phi_x\}_{x\in\mathfrak{g}}$ that make up the action on V annihilate a non-zero vector $v \in V$, because then one can obtain the desired result by induction on $\dim V$ (we can obtain a full flag of subspaces of V which is preserved by $\{\phi_x\}_{x\in\mathfrak{g}}$ by taking a full flag of subspaces of $V/\mathbb{K}v$ and appending $\mathbb{K}v$ to it). We will prove that all $\{\phi_x\}_{x\in\mathfrak{g}}$ annihilate a common non-zero vector by induction on $\dim \mathfrak{g}$. First of all, we may replace $\mathfrak{g}$ by the image of the action homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$, which allows us to assume $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. Then let us assume that

$$\mathfrak{g} \text{ has a codimension 1 ideal } \mathfrak{i} \tag{101}$$

By the induction hypothesis for $\mathfrak{i}$, the subspace

$$W = \left\{ v \in V \middle| x \cdot v = 0, \ \forall x \in \mathfrak{i} \right\}$$

is non-zero. Fix $y \in \mathfrak{g} \backslash \mathfrak{i}$. Because

$$x \cdot (y \cdot w) = y \cdot (x \cdot w) + \underbrace{[x, y]}_{\in \mathfrak{i}} \cdot w = 0$$

for all $x \in \mathfrak{i}$ and $w \in W$, we conclude that the action of y sends W to itself. However, the action of y is via a nilpotent operator. Since a nilpotent operator always annihilates some non-zero vector, we conclude that there exists $0 \neq w \in W$ such that $y \cdot w = 0$. As $x \cdot w = 0$ for all $x \in \mathfrak{i}$, we conclude that the action of all elements of $\mathfrak{g}$ annihilates w.

Let us now explain why there exists a codimension 1 ideal $\mathfrak{i} \subset \mathfrak{g}$, thus justifying the assumption (101). We choose $\mathfrak{i}$ to be maximal proper Lie subalgebra of $\mathfrak{g}$, and assume that codim $\mathfrak{i} > 1$. Consider the representation

$$\mathfrak{i} \curvearrowright \mathfrak{g}/\mathfrak{i}, \qquad x \cdot (y \bmod \mathfrak{i}) = ([x, y] \bmod \mathfrak{i})$$

The action above is by nilpotent operators, being a block of the action $\mathrm{ad} : \mathfrak{i} \curvearrowright \mathfrak{g}$, which is by nilpotent operators due to the assumption $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ (there is a subtlety to prove here: if $X \in \mathfrak{gl}(V)$ is a nilpotent operator, show that $\mathrm{ad}_X : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is nilpotent, i.e. $\mathrm{ad}_X^n = 0$ for some n). The inductive hypothesis of Theorem 12 implies that there exists $y \in \mathfrak{g} \backslash \mathfrak{i}$ such that $[\mathfrak{i}, y] \subseteq \mathfrak{i}$. This implies that $\mathfrak{i} \oplus \mathbb{K}y$ is a larger proper Lie subalgebra than $\mathfrak{i}$, which provides a contradiction. ☐

Theorem 12 implies the following result, commonly known as **Engel's theorem**.

Corollary 4. *A Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if* $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$ *is nilpotent for all* $x \in \mathfrak{g}$.

The "only if" implication is straightforward. For the "if" implication, Theorem 12 applied to the adjoint representation gives us a flag of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathfrak{g}$$

so that $\mathrm{ad}_x(V_i) \subseteq V_{i-1}$ for all i. Therefore, for any $x_1, \dots, x_{n+1} \in \mathfrak{g}$, we have

$$0 = \mathrm{ad}_{x_1} \circ \mathrm{ad}_{x_2} \circ \cdots \circ \mathrm{ad}_{x_n}(x_{n+1}) = [x_1, [x_2, \dots, [x_n, x_{n+1}] \dots]]$$

which precisely means that $\underbrace{[\mathfrak{g}, [\mathfrak{g}, \dots, [\mathfrak{g}, \mathfrak{g}] \dots]]}_{n+1 \text{ copies of } \mathfrak{g}} = 0$.

7.2

By analogy with the situation of groups, we say that a Lie algebra is **simple** if it has no proper ideals, and it is not abelian. The reason for the latter restriction is that it ensures that

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \tag{102}$$

as the ideal $[\mathfrak{g}, \mathfrak{g}]$ cannot be 0 if $\mathfrak{g}$ is not abelian. As a consequence, we have:

Corollary 5. *Any one-dimensional representation of a simple Lie algebra is 0.*

The following notion is a more general version of simplicity.

Definition 20. *A Lie algebra is called* ***semisimple*** *if it has no solvable ideals other than 0.*

Note that any simple Lie algebra is also semisimple. The reason is that its only ideals are 0 and $\mathfrak{g}$, but $\mathfrak{g}$ could not be solvable, because otherwise $[\mathfrak{g}, \mathfrak{g}]$ would be a proper ideal. The following notion, that of radical of a Lie algebra, measures how far a general Lie algebra is from being semisimple.

Proposition 11. *In any Lie algebra* $\mathfrak{g}$*, the sum of two solvable ideals (i.e. the smallest ideal containing the two) is solvable. Therefore, there exists a maximal solvable ideal called the* ***radical***

$$\boxed{\mathrm{rad}(\mathfrak{g}) \subseteq \mathfrak{g}}$$

and $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ *is semisimple.*

Proof. Let $\mathfrak{i}$ and $\mathfrak{j}$ be solvable ideals of $\mathfrak{g}$. The natural isomorphism

$$(\mathfrak{i} + \mathfrak{j})/\mathfrak{i} \cong \mathfrak{j}/(\mathfrak{i} \cap \mathfrak{j})$$

(itself an analogue of the second isomorphism theorem for Lie algebras) realizes $\mathfrak{i} + \mathfrak{j}$ as an extension of the solvable Lie algebras $\mathfrak{i}$ and a quotient of $\mathfrak{j}$. By Proposition 10, $\mathfrak{i} + \mathfrak{j}$ is solvable. Therefore, the maximal solvable ideal $\mathrm{rad}(\mathfrak{g})$ must be defined as the sum of all solvable ideals. The semisimplicity of $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ follows from the maximality of $\mathrm{rad}(\mathfrak{g})$ and the correspondence Theorem 6 (complete the argument just given).

$\square$

The definition of the radical entails the fact that any Lie algebra admits a short exact sequence

$$0 \to \mathrm{rad}(\mathfrak{g}) \xrightarrow{\iota} \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}_{ss} \to 0$$

where $\mathfrak{g}_{ss}$ is semisimple. In fact, an important result called **Levi's theorem** (which we will not have time to prove) states that the above short exact sequence actually splits if the ground field has characteristic 0, i.e.

$$\exists \psi : \mathfrak{g}_{ss} \to \mathfrak{g} \quad \text{s.t.} \quad \pi \circ \psi = \mathrm{Id}_{\mathfrak{g}_{ss}} \tag{103}$$

This means that $\mathfrak{g}_{ss}$ can be perceived as a subalgebra of $\mathfrak{g}$, although not as an ideal.

7.3

By generalizing Corollary 3, any irreducible representation (over an algebraically closed field)

$$\mathfrak{g} \curvearrowright V$$

has the property that $\mathrm{rad}(\mathfrak{g})$ acts by scalars. Proof: as in the proof of Theorem 11, the subspace

$$W = \left\{ v \in V \middle| x \cdot v = \lambda(x) v, \forall x \in \mathrm{rad}(\mathfrak{g}) \right\}$$

is non-zero for some linear functional λ on $\mathrm{rad}(\mathfrak{g})$. However, you can prove by analogy with Theorem 11 that any $y \in \mathfrak{g} \backslash \mathrm{rad}(\mathfrak{g})$ also sends W to W; because V is irreducible, then $V = W$. As the ideal

$$[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] \subseteq \mathfrak{g} \tag{104}$$

was just shown to act on irreducible representations by 0, it is customary to quotient out this ideal from $\mathfrak{g}$. This naturally leads to the following.

Definition 21. *A Lie algebra* $\mathfrak{g}$ *is called* ***reductive*** *if* $\mathrm{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.

The defining condition of a reductive Lie algebra is actually equivalent to $\mathrm{rad}(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g}) \Leftrightarrow [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = 0$, because the opposite inclusion $\mathfrak{z}(\mathfrak{g}) \subseteq \mathrm{rad}(\mathfrak{g})$ is true in any Lie algebra (prove this claim). With this in mind, Levi's theorem (103) implies that a reductive Lie algebra splits as

$$\mathfrak{g} \cong \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$$

In this case, $\mathfrak{g}_{ss}$ is actually an ideal of $\mathfrak{g}$, and the direct sum in the RHS is in the sense of Definition 15. Thus, we see that reductive Lie algebras are obtained from semisimple ones by adding a center. The main example is (52), in which the general linear Lie algebra (reductive) is the direct sum of the special linear Lie algebra (semisimple, as we will shortly see) and a one-dimensional center.

7.4

Semisimple and reductive Lie algebras can be described in terms of their bilinear forms, as per the following notion.

Definition 22. *If $\mathfrak{g}$ is a Lie algebra over a field $\mathbb{K}$, then a symmetric bilinear form*

$$\mathfrak{g} \times \mathfrak{g} \xrightarrow{(\cdot,\cdot)} \mathbb{K}$$

is called ***invariant*** *if*

$$([x,y],z) + ([x,z],y) = 0 \tag{105}$$

for all $x,y,z \in \mathfrak{g}$. We will write ***s.i.b.f.*** *for a symmetric invariant bilinear form.*

Condition (105) is nice because the orthogonal complement of an ideal with respect to a s.i.b.f. is also an ideal. The following is a great source of s.i.b.f.'s (we leave this claim as an exercise to you).

Proposition 12. *For any representation $\mathfrak{g} \curvearrowright V$, the assignment*

$$(x,y)_V = \mathrm{tr}_V\left(\phi_x \circ \phi_y\right) \tag{106}$$

is a s.i.b.f.

For example, if V is the usual n-dimensional representation of $\mathfrak{gl}_n$, then $(X,Y)_V = \mathrm{tr}(XY)$. For a general Lie algebra $\mathfrak{g}$, a special role is played by setting V to be the adjoint representation, in which case the s.i.b.f.

$$(x,y)_{\mathfrak{g}} = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_x \circ \mathrm{ad}_y) \tag{107}$$

is called the **Killing form**.

Theorem 13. *If the s.i.b.f.* (106) *is non-degenerate for some representation V, then $\mathfrak{g}$ is reductive.*

Proof. There exists a flag of subrepresentations

$$0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$$

with V_i/V_{i-1} irreducible for all $i \in \{1,\dots,k\}$. The fact that the matrices ϕ_x are all block upper triangular with respect to the flag above implies that

$$(x,y)_V = \sum_{i=1}^{k} (x,y)_{V_i/V_{i-1}}$$

for all $x,y \in \mathfrak{g}$. If we take $x \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$, then we have already seen in (104) that such x act by 0 in all irreducible representations, and thus lie in the kernel of $(\cdot,\cdot)_V$. Since the latter is assumed non-degenerate, this implies $x = 0$, as desired.

□

As a consequence of this Theorem, all the matrix Lie algebras that we encountered in this course ($\mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{o}_n, \mathfrak{sp}_{2n}, \mathfrak{u}(n), \mathfrak{su}(n)$) are reductive, which you can prove by showing that the s.i.b.f. given by their usual matrix representation is non-degenerate.

7.5

We have just seen that the s.i.b.f.'s (106) give a criterion for a Lie algebra being reductive. We will now see that the Killing form is even more powerful, as evidenced by the following result, typically called **Cartan's criterion of solvability/semisimplicity**, which holds over any field of characteristic 0.

Theorem 14. *(a) A Lie algebra $\mathfrak{g}$ is solvable if and only if*

$$([x, y], z)_{\mathfrak{g}} = 0, \quad \forall x, y, z \in \mathfrak{g}$$

(b) A Lie algebra is semisimple if and only if its Killing form is non-degenerate.

We will prove Theorem 14 using certain tools that will be developed in next lecture. But let us apply it to $\mathfrak{gl}_n = (\text{scalar matrices}) \oplus \mathfrak{sl}_n$: clearly scalar matrices are in the kernel of the Killing form, which shows that $\mathfrak{gl}_n$ is not semisimple. On the other hand, if we let V be the usual n-dimensional representation of $\mathfrak{gl}_n$, we have

$$(E_{ij}, E_{i'j'})_V = \mathrm{tr}_V(E_{ij}E_{i'j'}) = \delta_{i'j}\delta_{ij'}$$

which is clearly non-degenerate. Thus, we conclude that $\mathfrak{gl}_n$ is reductive. The bilinear form above is also non-degenerate for $\mathfrak{sl}_n$, but because the latter Lie algebra is simple, the following result also implies the non-degeneracy of the Killing form of $\mathfrak{sl}_n$.

Lemma 2. *Any two non-degenerate s.i.b.f.'s on a simple Lie algebra (over an algebraically closed field) are proportional.*

Proof. It is easy to check that any non-degenerate s.i.b.f. on a Lie algebra $\mathfrak{g}$ gives an isomorphism

$$\boxed{\mathfrak{g} \cong \mathfrak{g}^*}$$

of representations of the Lie algebra $\mathfrak{g}$ (where the LHS is the adjoint representation and the RHS is the dual of the adjoint representation). The fact that $\mathfrak{g}$ is simple is equivalent to the adjoint representation being irreducible, and then the fact that any two isomorphisms $\mathfrak{g} \cong \mathfrak{g}^*$ differ by a constant multiple is a consequence of Schur's Lemma.

□

7.6

Let us give without proof some important results on Killing forms of real Lie algebras, see Lecture 3.

Theorem 15. *(a) For any compact real Lie group G, its Lie algebra $\mathfrak{g}$ is reductive and its Killing form is negative-semidefinite (the kernel of the form is just $\mathfrak{z}(\mathfrak{g})$).*

(b) As a partial converse, if $\mathfrak{g}$ is a real Lie algebra with a negative-definite Killing form, then any connected real Lie group with Lie algebra $\mathfrak{g}$ is compact.

The situation of real Lie algebras with positive-definite Killing form is much simpler and less interesting than the above Theorem: there are no such Lie algebras except 0. To see this, let $\mathfrak{g}$ be a real Lie algebra, and pick an orthonormal basis

$$\mathfrak{g} = \bigoplus_{i=1}^{n} \mathbb{R}x_i$$

for the Killing form. If we let γ_{ij}^k be the structure constants for the Lie bracket, i.e.

$$[x_i, x_j] = \mathrm{ad}_{x_i}(x_j) = \sum_k \gamma_{ij}^k x_k$$

then we have for all $i \in \{1, \dots, n\}$

$$(x_i, x_i)_{\mathfrak{g}} = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_{x_i} \circ \mathrm{ad}_{x_i}) = \sum_{j,k=1}^{n} \gamma_{ij}^k \gamma_{ik}^j \tag{108}$$

However, the identity

$$\gamma_{ij}^k = ([x_i, x_j], x_k)_{\mathfrak{g}} = -([x_i, x_k], x_j)_{\mathfrak{g}} = -\gamma_{ik}^j \tag{109}$$

implies that the right-hand side of (108) is non-positive, which contradicts the positive-definiteness of the Killing form.

Lecture 8

8.1

Semisimple Lie algebras have two important features: complete reducibility of representations, and Jordan decompositions, both of which we will now state and prove. The ground field $\mathbb{K}$ must have characteristic 0 throughout the entire lecture.

Theorem 16. *If V is any finite-dimensional representation of a semisimple Lie algebra $\mathfrak{g}$, then for any subrepresentation $W \subseteq V$ there exists another subrepresentation $W' \subseteq V$ such that*

$$V = W \oplus W' \tag{110}$$

*After repeated applications of this result, we conclude that any representation of a semisimple Lie algebra is a direct sum of irreducible representations, a phenomenon called **complete reducibility**.*

The semisimplicity assumption is key, as we have already seen in (41) an example where complete reducibility fails. In fact, that example failed because the action was given in terms of matrices which are not diagonalizable, which leads us into a discussion of Jordan decompositions. Let us start with the following result, which we leave to you.

Proposition 13. *A linear transformation $f : V \to V$ is called **semisimple** if for any subspace $W \subseteq V$ preserved by f there exists another subspace preserved by f such that*

$$V = W \oplus W'$$

If V is a finite-dimensional vector space over an algebraically closed field $\mathbb{K}$, then f is semisimple if and only if f is diagonalizable.

The well-known Jordan decomposition basically says that any linear transformation f on a finite-dimensional vector space can be uniquely decomposed as

$$f = f_{ss} + f_n \tag{111}$$

where f_{ss} is semisimple, f_n is nilpotent, and $f_{ss}f_n = f_n f_{ss}$. Moreover, f_{ss} and f_n are polynomials in f with zero constant term. We will soon show that the decomposition (111) extends uniformly from individual linear transformations to elements of semisimple Lie algebras, as follows.

Theorem 17. *If $\mathfrak{g}$ is a semisimple Lie algebra, then any $x \in \mathfrak{g}$ admits a unique decomposition*

$$x = x_{ss} + x_n \tag{112}$$

such that $[x_{ss}, x_n] = 0$, and x_{ss} (respectively x_n) acts as a semisimple (respectively nilpotent) operator in any representation of $\mathfrak{g}$. This is true in particular in the adjoint representation, so

$$(\mathrm{ad}_x)_{ss} = \mathrm{ad}_{x_{ss}} \quad \text{and} \quad (\mathrm{ad}_x)_n = \mathrm{ad}_{x_n} \tag{113}$$

*One calls (112) the **abstract** Jordan decomposition.*

Formula (113) is very strong. For example, we claim that it implies that

$$[x, y] = 0 \quad \Leftrightarrow \quad [x_{ss}, y] = [x_n, y] = 0 \tag{114}$$

for any $x, y \in \mathfrak{g}$. To see this, the fact that $(\mathrm{ad}_x)_{ss} = \mathrm{ad}_{x_{ss}}$ implies that the latter operator is a polynomial in ad_x with zero constant term. If $[x, y] = 0$, then the aforementioned polynomial annihilates y, which implies that $[x_{ss}, y] = 0$ (hence also $[x_n, y] = 0$).

8.2

In the remainder of this Lecture, we will prove the big results that were simply stated above. For technical reasons, we will start with the Cartan criteria of solvability and semisimplicity.

Proof. of Theorem 14: (a) Since a Lie algebra over $\mathbb{K}$ is solvable if and only if its extension over the algebraic closure of $\mathbb{K}$ is solvable (the property of (93) terminating with 0 is unchanged by field extension), then we will assume the ground field is algebraically closed. In this case, the "only if" statement is an easy case of Lie's theorem 11 for the adjoint representation: indeed, the fact that ad_z is upper triangular for all $z \in \mathfrak{g}$ implies that $\mathrm{ad}_{[x,y]}$ is strictly upper triangular for all $x, y \in \mathfrak{g}$. Therefore, $([x, y], z)_{\mathfrak{g}}$ is the trace of a strictly upper triangular matrix, and thus equal to 0.

Lemma 3. *If a Lie subalgebra $\mathfrak{a} \subseteq \mathfrak{gl}_n$ has the property that*

$$\mathrm{tr}(xy) = 0, \quad \forall x, y \in \mathfrak{a} \tag{115}$$

then $\mathfrak{a}$ is solvable.

Let us first deduce the "if" statement of Theorem 14.a) from Lemma 3. By Proposition 10, it suffices to show that $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is solvable. So by replacing $\mathfrak{g}$ with $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, we may assume that the adjoint representation provides an injection

$$\mathfrak{g} \hookrightarrow \mathrm{End}(\mathfrak{g})$$

Therefore, the Lie subalgebra $\mathfrak{a} = [\mathfrak{g}, \mathfrak{g}]$ satisfies the hypotheses of Lemma 3, and is thus solvable. Since $\mathfrak{g}/\mathfrak{a}$ is abelian and thus solvable, then Proposition 10 implies that $\mathfrak{g}$ is solvable.

To prove Lemma 3, it suffices to show that $[x, y]$ is a nilpotent matrix for all $x, y \in \mathfrak{a}$, because then the first blue claim of Lecture 7 and Corollary 4 would imply that $[\mathfrak{a}, \mathfrak{a}]$ is nilpotent (hence solvable, hence Proposition 10 would imply that $\mathfrak{a}$ is solvable). Thus, let us pick arbitrary $x, y \in \mathfrak{a}$ and assume that the eigenvalues of $[x, y]$ are $\lambda_1, \dots, \lambda_n$ counted with multiplicities (we are still working over the algebraic closure of the ground field). Our goal is to show that $\lambda_1 = \cdots = \lambda_n = 0$, so assume that at least one of the λ_i's is non-zero. Then show that there exists a $\mathbb{Q}$-linear functional

$$\zeta : \mathrm{span}_{\mathbb{Q}}(\lambda_1, \dots, \lambda_n) \to \mathbb{Q}$$

such that

$$\sum_{i=1}^{n} \lambda_i \zeta(\lambda_i) \neq 0 \tag{116}$$

(if this ζ seems strange to you, then just assume we are working over $\mathbb{C}$ and we replace $\zeta(\lambda_i)$ by $\overline{\lambda_i}$ from now on). Because ζ is linear, there exists an interpolation polynomial $P(t)$ such that

$$P(\lambda_i - \lambda_j) = \zeta(\lambda_i) - \zeta(\lambda_j) \tag{117}$$

for all $i, j \in \{1, \dots, n\}$. Let

$$A = [x, y]_{ss} = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \quad \Rightarrow \quad \mathrm{ad}_A = \mathrm{diag}(\lambda_i - \lambda_j)_{1 \leq i,j \leq n}$$

and $B = \mathrm{diag}(\zeta(\lambda_1), \dots, \zeta(\lambda_n))$. Therefore, (117) implies that

$$\mathrm{ad}_B = \mathrm{diag}(\zeta(\lambda_i) - \zeta(\lambda_j))_{1 \leq i,j \leq n} = P(\mathrm{ad}_A)$$

Moreover, as $\mathrm{ad}_A = \mathrm{ad}_{[x,y]_{ss}} = \left(\mathrm{ad}_{[x,y]}\right)_{ss}$, we conclude that ad_A is itself a polynomial in $\mathrm{ad}_{[x,y]}$. This implies that $\mathrm{ad}_B = Q(\mathrm{ad}_{[x,y]})$ for some polynomial $Q(t)$, and thus

$$\mathrm{ad}_B(\mathfrak{a}) = Q(\mathrm{ad}_{[x,y]})(\mathfrak{a}) \subseteq \mathfrak{a}$$

With this in mind, we have for all $x, y \in \mathfrak{a}$

$$\sum_{i=1}^{n} \lambda_i \zeta(\lambda_i) = \mathrm{tr}(AB) = \mathrm{tr}([x,y]B) = \mathrm{tr}(\mathrm{ad}_B(x)y) = 0$$

with the last equality being precisely the hypothesis (115). We have therefore contradicted (116).

(b) For the "only if" statement, let us consider the kernel $\mathfrak{i}$ of the Killing form of a semisimple Lie algebra $\mathfrak{g}$. By (105), $\mathfrak{i}$ is an ideal of $\mathfrak{g}$. Moreover, for any $x, y, z \in \mathfrak{i}$

$$\mathrm{ad}_{[x,y]} \circ \mathrm{ad}_z$$

calculated in $\mathfrak{g}$ is a block matrix, with one of the diagonal blocks being given by the same composition but calculated in $\mathfrak{i}$, and the other diagonal block being 0. Thus, the fact that $([x,y],z)_{\mathfrak{g}} = 0$ implies that $([x,y],z)_{\mathfrak{i}} = 0$, for all $x, y, z \in \mathfrak{i}$. Part (a) implies that $\mathfrak{i}$ is solvable, and by the definition of semisimplicity we conclude that $\mathfrak{i} = 0$ (i.e. the Killing form is non-degenerate).

Let us now prove the "if" statement, which relies on the following claim (which we leave to you, using the fact the last non-zero term in the derived series of $\mathrm{rad}(\mathfrak{g})$ would be an abelian ideal of $\mathfrak{g}$).

Lemma 4. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if it has no non-zero abelian ideals.*

Since any abelian ideal $\mathfrak{i}$ of a Lie algebra $\mathfrak{g}$ would lie in the kernel of the Killing form (because ad_x sends $\mathfrak{g}$ to $\mathfrak{i}$ and $\mathfrak{i}$ to 0 for all $x \in \mathfrak{i}$), then the non-degeneracy of the Killing form implies the non-existence of abelian ideals other than 0. □

8.3

Cartan's criterion for semisimplicity has a number of interesting consequences. For one thing, a real Lie algebra is semisimple if and only if its complexification is semisimple (note that this does not hold for simple Lie algebras). More important is the following characterization of Lie algebras.

Lemma 5. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if it is a direct sum of simple Lie algebras.*

Proof. We already showed that simple Lie algebras are semisimple, and so their Killing forms are non-degenerate by Theorem 14. Since the Killing form of a direct sum of Lie algebras is the sum of the Killing forms of its constituents, we conclude that any direct sum of simple algebras has non-degenerate Killing form, hence is semisimple by Theorem 14.

For the opposite direction, consider a semisimple Lie algebra $\mathfrak{g}$ which is not simple. Therefore, if has a proper ideal $\mathfrak{i} \subset \mathfrak{g}$. If we let $\mathfrak{j}$ be the complement of $\mathfrak{i}$ with respect to the Killing form, then show that the non-degeneracy of the Killing form implies that

$$\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{j}$$

and the invariance of the Killing form implies that $\mathfrak{j}$ is an ideal. Therefore, the decomposition above is a direct sum of Lie algebras, in the sense of Definition 15. Since $\mathfrak{i}$ and $\mathfrak{j}$ are semisimple Lie algebras in turn, we can repeat this algorithm by induction on $\dim \mathfrak{g}$.

□

As a consequence of Lemma 5, we have that

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \tag{118}$$

for any semisimple Lie algebra $\mathfrak{g}$, because the property above holds for simple Lie algebras (we saw this in (102)) and it is preserved under direct sums. Therefore, we conclude the following.

Corollary 6. *Any one-dimensional representation of a semisimple Lie algebra is 0.*

8.4

We will now prove Theorem 16 on complete reducibility. The following notion is key.

Proposition 14. *For any non-degenerate s.i.b.f. $(\cdot, \cdot)$ on a Lie algebra $\mathfrak{g}$, its Casimir element is defined as*

$$C = \sum_i x_i \otimes x^i \in U\mathfrak{g} \tag{119}$$

where $\{x_i\}, \{x^i\}$ run over dual bases of $\mathfrak{g}$ with respect to the s.i.b.f. Then $C \in Z(U\mathfrak{g})$.

By basic algebra, show that C does not depend on the choice of dual bases $\{x_i\}, \{x^i\}$. If $(\cdot, \cdot)_{\mathfrak{g}}$ is the Killing form, then the corresponding $C_{\mathfrak{g}}$ is usually called <u>the</u> Casimir element of $\mathfrak{g}$. The potential abuse of terminology is mild for a simple Lie algebra, because Lemma 2 implies that any two Casimir elements are proportional. Because any representation $\mathfrak{g} \curvearrowright V$ is also a representation of $U\mathfrak{g}$, the Casimir element acts in V; we have already encountered this operator for $\mathfrak{sl}_2$ in Subsection 5.5.

Proof. of Proposition 14: Let us assume $x_i = x^i$ is an orthonormal basis with respect to the given symmetric invariant bilinear form, so $C = \sum_i x_i \otimes x_i$. If we let

$$[x_i, x_j] = \sum_k \gamma_{ij}^k x_k$$

then we have $\gamma_{ik}^j = -\gamma_{ki}^j \overset{(105),(109)}{=} \gamma_{kj}^i = -\gamma_{jk}^i$. Therefore, we have for any k

$$C \otimes x_k - x_k \otimes C = \sum_i \Big([x_i, x_k] \otimes x_i + x_i \otimes [x_i, x_k]\Big) = \sum_{i,j} \gamma_{ik}^j \Big(x_j \otimes x_i + x_i \otimes x_j\Big) = 0$$

□

8.5

Because any Casimir element is central, Schur's Lemma implies that it acts by a constant in any irreducible representation. More specifically, for an irreducible representation $\mathfrak{g} \curvearrowright V$, the Casimir element defined with respect to the symmetric invariant bilinear form (106) acts by the constant

$$\frac{\dim \mathfrak{g}}{\dim V} \tag{120}$$

(prove this under the assumption that the form $(\cdot, \cdot)_V$ is non-degenerate; if on the other hand this form has a non-trivial kernel $\mathfrak{i} \subseteq \mathfrak{g}$, then we simply replace $\mathfrak{g}$ by $\mathfrak{g}/\mathfrak{i}$ in (119)). Even more generally, it is a fact that <u>any</u> Casimir element of a semisimple Lie algebra acts by a non-zero scalar in <u>any</u> irreducible finite-dimensional representation, but we will not need this.

Proof. of Theorem 16: Let us first consider the case when $W \subset V$ has codimension 1. If W is irreducible, then the discussion above shows that there is a Casimir element C which acts by a non-zero constant on W. By Corollary 6, the action of $\mathfrak{g}$ on the one-dimensional quotient V/W is 0, so we conclude that C sends V to W. Thus, $W' = \text{Ker } C|_V$ is non-empty, and because $C|_W$ is a non-zero scalar, we conclude that W' is one-dimensional and complementary to W. The fact that W' is a subrepresentation follows from

$$C \cdot (x \cdot v) = x \cdot (C \cdot v) \quad \Rightarrow \quad x \cdot W' \subseteq W'$$

for all $x \in \mathfrak{g}$, where the first equality is due to the fact that C is central. Having proved the complete reducibility when W is irreducible (but still codimension 1 inside V), let us now prove the case of general W (but still codimension 1 inside V) by induction on dim V. If W is not irreducible, then we may consider a maximal proper subrepresentation $\bar{W} \subsetneq W$ (which exists due to finite-dimensionality) and simply run the discussion above for the representation $\mathfrak{g} \curvearrowright V/\bar{W}$ and its codimension 1 subrepresentation $W/\bar{W}$. We obtain a subrepresentation $\bar{W} \subset \bar{W}' \subset V$ such that

$$V/\bar{W} = W/\bar{W} \oplus \bar{W}'/\bar{W}$$

with $\bar{W}$ having codimension 1 inside $\bar{W}'$. Repeating the argument above gives us a decomposition $\bar{W}' = \bar{W} \oplus W'$ for some one-dimensional subrepresentation $W' \subset \bar{W}'$, and so $V = W \oplus W'$ is the required decomposition in (110).

Having proved the Theorem for any subrepresentation W of codimension 1 in V, let us consider an arbitrary subrepresentation $W \subset V$ and define

$$\widetilde{V} = \left\{ f : V \to W \middle| f|_W = \text{scalar} \right\}$$
$$\widetilde{W} = \left\{ f : V \to W \middle| f|_W = 0 \right\}$$

We may make $\widetilde{V}$ into a representation of $\mathfrak{g}$ via $(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$, and it is clear that

$$x \cdot \widetilde{V} = \widetilde{W}$$

for all $x \in \mathfrak{g}$. As $\widetilde{W}$ has codimension 1 inside $\widetilde{V}$, the first part of this proof implies the existence of

$$f \in \widetilde{V} \backslash \widetilde{W}, \quad \text{s.t. } x \cdot f = \lambda(x) f, \ \forall x \in \mathfrak{g}$$

where $\lambda : \mathfrak{g} \to \mathbb{K}$ is some linear functional. Then, $f|_W = \alpha \cdot \text{Id}_W$ for some non-zero α, so $W' = \text{Ker } f$ is a complementary subspace to W inside V. Moreover, W' is a subrepresentation because

$$f(v) = 0 \quad \Rightarrow \quad f(x \cdot v) = x \cdot f(v) - (x \cdot f)(v) = x \cdot f(v) - \lambda(x) f(v) = 0$$

$\square$

8.6

Let us now prove Theorem 17 on the abstract Jordan decomposition in semisimple Lie algebras.

Proof. of Theorem 17: We begin by claiming that for any semisimple subalgebra

$$\mathfrak{g} \subseteq \mathfrak{gl}(V)$$

the semisimple and nilpotent parts of any element $x \in \mathfrak{g}$ (regarded as linear transformations $V \to V$, see (111)) also lie in $\mathfrak{g}$. We will prove this fact later, but let us apply it for the adjoint representation

$$\mathfrak{g} \hookrightarrow \mathrm{End}(\mathfrak{g})$$

(which is faithful since semisimple Lie algebras have trivial center). Then for any $x \in \mathfrak{g}$ there exists a decomposition

$$x = x_{ss} + x_n \tag{121}$$

with $x_{ss}, x_n \in \mathfrak{g}$ such that (113) holds. Let us now consider an arbitrary representation $\mathfrak{g} \curvearrowright V$ and the associated Lie algebra homomorphism

$$\mathfrak{g} \xrightarrow{\phi} \mathfrak{gl}(V)$$

We want to show that for any $x \in \mathfrak{g}$, we have

$$\phi(x)_{ss} = \phi(x_{ss}) \quad \text{and} \quad \phi(x)_n = \phi(x_n)$$

i.e. the abstract Jordan decomposition gives rise to the usual Jordan decomposition in V. By replacing $\mathfrak{g}$ with Im ϕ (which is also semisimple on account of it being a quotient of a semisimple Lie algebra) we may regard $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}(V)$. Then any $x \in \mathfrak{g}$ admits a Jordan decomposition in the context of linear transformations $V \to V$

$$x = x'_{ss} + x'_n \tag{122}$$

where the claim at the beginning of the proof establishes the fact that $x'_{ss}, x'_n \in \mathfrak{g}$. You have already shown last week that if $y \in \mathfrak{gl}(V)$ is nilpotent, then ad_y is nilpotent; it is also true that if $y \in \mathfrak{gl}(V)$ is semisimple, then ad_y is semisimple (one of the exercises on this week's exercise sheet will essentially prove this over an algebraically closed field). Therefore,

$$\mathrm{ad}_x = \mathrm{ad}_{x'_{ss}} + \mathrm{ad}_{x'_n} \tag{123}$$

is a Jordan decomposition in $\mathrm{End}(\mathfrak{gl}(V))$. Since the semisimple and nilpotent parts of any operator are polynomials in said operator, then $\mathrm{ad}_{x'_{ss}}$ and $\mathrm{ad}_{x'_n}$ send $\mathfrak{g}$ to $\mathfrak{g}$. It is easy to see that a diagonal block of a semisimple/nilpotent linear transformation is also semisimple/nilpotent, and so the restriction of (123) to $\mathfrak{g}$ is also a Jordan decomposition. By uniqueness of the latter, we conclude that $\mathrm{ad}_{x'_{ss}} = \mathrm{ad}_{x_{ss}}$ and $\mathrm{ad}_{x'_n} = \mathrm{ad}_{x_n}$ as linear transformations of $\mathfrak{g}$. Therefore, the faithfulness of the adjoint representation of a semisimple Lie algebra implies that the decompositions (121) and (122) coincide, i.e. the abstract Jordan decomposition agrees with the Jordan decomposition in V.

To prove the claim at the beginning of this proof, consider the following Lie subalgebras of $\mathfrak{gl}(V)$

- $\mathfrak{a} = \left\{ f : V \to V \middle| [f, \mathfrak{g}] \subseteq \mathfrak{g} \right\}$
- $\mathfrak{b}_W = \left\{ f : V \to V \middle| f(W) \subseteq W \text{ and } \mathrm{tr}(f|_W) = 0 \right\}$

for any $\mathfrak{g}$-invariant subspace $W \subseteq V$. It is almost obvious to see that

$$\mathfrak{g} \subseteq \mathfrak{g}' := \mathfrak{a} \bigcap_{\mathfrak{g}\text{-invariant } W \subseteq V} \mathfrak{b}_W$$

(the fact that any $x \in \mathfrak{g}$ acts on any $\mathfrak{g}$-invariant subspace $W \subseteq V$ by a traceless matrix is due to the fact that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ for a semisimple Lie algebra $\mathfrak{g}$). Consider the Jordan decomposition (111)

$$x = x_{ss} + x_n$$

of any $x \in \mathfrak{g}$, where $x_{ss}, x_n : V \to V$. Because x_{ss} and x_n are polynomial expressions in x, we have

$$x_{ss}, x_n \in \mathfrak{b}_W$$

for any $\mathfrak{g}$-invariant $W \subseteq V$ (a little thought is needed to see that x_{ss} and x_n are traceless on W). Moreover, as we showed in the previous paragraph, $\text{ad}_x = \text{ad}_{x_{ss}} + \text{ad}_{x_n}$ is a Jordan decomposition in $\text{End}(\mathfrak{gl}(V))$. Therefore, $\text{ad}_{x_{ss}}$ and ad_{x_n} are polynomial expressions in ad_x, which implies that

$$x_{ss}, x_n \in \mathfrak{a}$$

Putting the above two displays together implies that

$$x_{ss}, x_n \in \mathfrak{g}' \tag{124}$$

By the complete reducibility Theorem 16, the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}'$ decomposes as

$$\mathfrak{g}' = \mathfrak{g} \oplus S$$

for some $S \subseteq \mathfrak{gl}(V)$. Because $\mathfrak{g}$ is tautologically an ideal of $\mathfrak{a}$, we conclude that $\mathfrak{g}$ is an ideal of $\mathfrak{g}'$, and so $[\mathfrak{g}, S] = 0$. If we decompose V as a direct sum of irreducible representations of $\mathfrak{g}$

$$V = V_1 \oplus \cdots \oplus V_k$$

then any $f \in S$ is on one hand a $\mathfrak{g}$-intertwiner, while on the other hand f preserves each V_i and acts tracelessly on it. By Schur's lemma, the only option is that any $f \in S$ is actually 0 and so $\mathfrak{g}' = \mathfrak{g}$. Then (124) implies the claim at the beginning of this proof. $\square$

8.7

Bonus material (will not feature on the exam): the Jordan decomposition in the adjoint representation (113) can also be constructed as follows.

Definition 23. *A* ***derivation*** *of a Lie algebra $\mathfrak{g}$ is a linear transformation*

$$\zeta : \mathfrak{g} \to \mathfrak{g}$$

which satisfies the following version of the Leibniz rule

$$\zeta([y, z]) = [\zeta(y), z] + [y, \zeta(z)]$$

for all $y, z \in \mathfrak{g}$.

For any $x \in \mathfrak{g}$, the Jacobi identity implies that

$$\xi_x(y) = [x, y] \tag{125}$$

is a derivation. Such derivations are called **inner**, and any other derivation is called **outer**.

Lemma 6. *A semisimple Lie algebra only has inner derivations.*

Proof. The set $\mathrm{Der}(\mathfrak{g})$ of derivations of $\mathfrak{g}$ is itself a Lie algebra, with respect to the Lie bracket

$$[\zeta, \zeta'](y) = \zeta(\zeta'(y)) - \zeta'(\zeta(y))$$

and it is easy to see that the function

$$\mathfrak{g} \to \mathrm{Der}(\mathfrak{g}), \quad x \rightsquigarrow \xi_x \tag{126}$$

is a Lie algebra homomorphism. Because a semisimple Lie algebra $\mathfrak{g}$ has trivial center, the function above is injective. The fact that

$$[\xi_x, \zeta] = \xi_{\zeta(x)}, \quad \forall \zeta \in \mathrm{Der}(\mathfrak{g}), x \in \mathfrak{g} \tag{127}$$

implies that the injection (126) identifies $\mathfrak{g}$ with an ideal of $\mathrm{Der}(\mathfrak{g})$. Moreover, this injection identifies the Killing form on $\mathfrak{g}$ with the one on $\mathrm{Der}(\mathfrak{g})$. The non-degeneracy of the Killing form on $\mathfrak{g}$ means that we have a direct sum decomposition

$$\mathrm{Der}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{i}$$

(as in the proof of Lemma 5), where $\mathfrak{i}$ is an ideal of $\mathrm{Der}(\mathfrak{g})$. Because $[\mathfrak{g}, \mathfrak{i}] = 0$ as $\mathfrak{g}$ and $\mathfrak{i}$ are complementary ideals in $\mathrm{Der}(\mathfrak{g})$, for any $\zeta \in \mathfrak{i}$ we have by (127)

$$\xi_{\zeta(x)} = 0, \forall x \in \mathfrak{g} \quad \Rightarrow \quad \zeta(x) = 0, \forall x \in \mathfrak{g}$$

This shows that $\mathfrak{i} = 0$, and so every derivation is inner.

□

Let us now consider a Lie algebra $\mathfrak{g}$ over an algebraically closed ground field $\mathbb{K}$, and establish (113). For any $x \in \mathfrak{g}$, take the generalized eigenspace decomposition of the operator $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{\gamma \in \mathbb{K}} \mathfrak{g}_\gamma$$

where

$$\mathfrak{g}_\gamma = \left\{ y \in \mathfrak{g} \middle| (\mathrm{ad}_x - \gamma \cdot \mathrm{Id}_{\mathfrak{g}})^N (y) = 0 \text{ for } N \gg 0 \right\}$$

Define $(\mathrm{ad}_x)_{ss}$ as the operator which acts on $\mathfrak{g}_\gamma$ as multiplication by γ. We claim that

$$\zeta : \mathfrak{g} \to \mathfrak{g}, \quad \zeta(y) = (\mathrm{ad}_x)_{ss}(y) \tag{128}$$

is a derivation. To see this, take any $y \in \mathfrak{g}_\gamma$ and $z \in \mathfrak{g}_\delta$, and note that

$$(\mathrm{ad}_x - (\gamma + \delta) \cdot \mathrm{Id}_{\mathfrak{g}}) ([y, z]) = [(\mathrm{ad}_x - \gamma \cdot \mathrm{Id}_{\mathfrak{g}})(y), z] + [y, (\mathrm{ad}_x - \delta \cdot \mathrm{Id}_{\mathfrak{g}})(z)] \quad \Rightarrow$$

$$\Rightarrow \quad (\mathrm{ad}_x - (\gamma+\delta)\cdot \mathrm{Id}_{\mathfrak{g}})^N([y,z]) = \sum_{N_1+N_2=N} [(\mathrm{ad}_x - \gamma\cdot\mathrm{Id}_{\mathfrak{g}})^{N_1}(y), (\mathrm{ad}_x - \delta\cdot\mathrm{Id}_{\mathfrak{g}})^{N_2}(z)] = 0$$

if N is large enough. Thus, we conclude that $[y,z] \in \mathfrak{g}_{\gamma+\delta}$, which immediately implies that (128) is a derivation. By Lemma 6, $\zeta = \xi_{x_{ss}}$ for some $x_{ss} \in \mathfrak{g}$ and thus

$$(\mathrm{ad}_x)_{ss} = \mathrm{ad}_{x_{ss}} \tag{129}$$

If we let $x_n = x - x_{ss}$, we conclude that

$$(\mathrm{ad}_x)_n = \mathrm{ad}_{x_n} \tag{130}$$

The fact that x_{ss} and x_n commute follows from the fact that

$$0 = [(\mathrm{ad}_x)_{ss}, (\mathrm{ad}_x)_n] = [\mathrm{ad}_{x_{ss}}, \mathrm{ad}_{x_n}] = \mathrm{ad}_{[x_{ss},x_n]} \quad \Rightarrow \quad 0 = [x_{ss}, x_n]$$

The last implication is due to the fact that semisimple representations have zero kernel, and thus their adjoint representations are faithful.

Lecture 9

9.1

Having developed foundational results on semisimple Lie algebras, we will now describe them explicitly. Throughout the present section, we assume that the ground field is $\mathbb{C}$.

Definition 24. *Let $\mathfrak{g}$ be a semisimple Lie algebra. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called* ***toral*** *if it is abelian and consists of semisimple elements.*

The standard example of a toral subalgebra is any subspace of the set of diagonal matrices inside $\mathfrak{sl}_n$. Of course, one can replace the set of diagonal matrices by any conjugate thereof, which would lead to many more toral subalgebras. Since it is abelian, any toral subalgebra is isomorphic to $\mathbb{C}^{\oplus n}$ for some n. The space of $\mathbb{C}$-linear functions

$$\mathfrak{h}^* = \Big\{ \lambda : \mathfrak{h} \to \mathbb{C} \Big\}$$

has the same dimension as $\mathfrak{h}$. Throughout the present lecture, $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ denotes an arbitrary symmetric invariant bilinear form, which is non-degenerate (for example the Killing form).

Proposition 15. *For any semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, and any toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we have a decomposition*

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda \tag{131}$$

where

$$\boxed{\mathfrak{g}_\lambda = \Big\{ x \in \mathfrak{g} \Big| [h, x] = \lambda(h) x, \ \forall h \in \mathfrak{h} \Big\}} \tag{132}$$

Then we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \tag{133}$$

for all $\alpha, \beta \in \mathfrak{h}^*$, and the non-degenerate form $(\cdot, \cdot)$ has the property that

$$\mathfrak{g}_\alpha \times \mathfrak{g}_\beta \xrightarrow{(\cdot,\cdot)} \mathbb{C} \tag{134}$$

is non-degenerate if $\alpha + \beta = 0$ and is 0 if $\alpha + \beta \neq 0$.

Proof. The subspaces (131) are the joint eigenspaces of the commuting operators $\{\mathrm{ad}_x\}_{x \in \mathfrak{h}}$ on $\mathfrak{g}$ (these operators are all semisimple by assumption, hence simultaneously diagonalizable since we are working over $\mathbb{C}$). Property (133) is an immediate consequence of the Jacobi identity

$$[h, [x, y]] = [x, [h, y]] + [[h, x], y]$$

(show that the Jacobi identity is equivalent to the above formula) so if $x \in \mathfrak{g}_\alpha \Rightarrow [h, x] = \alpha(h) x$ and $y \in \mathfrak{g}_\beta \Rightarrow [h, y] = \beta(h) y$, then $[h, [x, y]] = (\alpha(h) + \beta(h))[x, y] \Rightarrow [x, y] \in \mathfrak{g}_{\alpha+\beta}$. As far as the restricted pairing (134) is concerned, the invariance implies that

$$\alpha(h)(x, y) = ([h, x], y) = -(x, [h, y]) = -\beta(h)(x, y)$$

for any $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$. This shows that $(x, y) \neq 0$ only if $\alpha + \beta = 0$. The fact that the case $\alpha + \beta = 0$ of the restricted pairing (134) is non-degenerate then follows from the overall non-degeneracy of the pairing $(\cdot, \cdot)$.

□

9.2

In particular, Theorem 13 applied to the non-degenerate pairing (134) for $\alpha = \beta = 0$ implies that $\mathfrak{g}_0$ is reductive. Since $\mathfrak{h}$ is abelian, we have $\mathfrak{g}_0 \supseteq \mathfrak{h}$. The following Definition pertains to the opposite inclusion.

Definition 25. *A toral subalgebra $\mathfrak{h}$ of a complex semisimple Lie algebra $\mathfrak{g}$ is called a* ***Cartan subalgebra*** *if*

$$[x, \mathfrak{h}] \subseteq \mathfrak{h} \quad \Rightarrow \quad x \in \mathfrak{h} \tag{135}$$

Note that (135) implies that $\mathfrak{g}_0 = \mathfrak{h}$. The standard example of a Cartan subalgebra of $\mathfrak{sl}_n$ is the set of all diagonal traceless matrices, or any conjugate thereof. From this example, we see that what distinguishes Cartan subalgebras among toral subalgebras is the fact that they are maximal (i.e. the subspace of all diagonal matrices versus some subspace of diagonal matrices). In fact, this is a completely general phenomenon, as we will now see.

Proposition 16. *For a complex semisimple Lie algebra $\mathfrak{g}$, a Cartan subalgebra is the same thing as a maximal toral subalgebra.*

Proof. The fact that a Cartan subagebra is maximal easily follows from (135), since toral subalgebras are by their very nature abelian. For the converse, let us consider a maximal toral subalgebra $\mathfrak{h}$ and prove that

$$\mathfrak{g}_0 = \mathfrak{h} \tag{136}$$

Once we do so, it will follow that $\mathfrak{h}$ is a Cartan subalgebra, as it satisfies the defining property (135) (because of (133), if a certain x has a non-zero component in some $\mathfrak{g}_\alpha$ with $\alpha \neq 0$, then $[x, \mathfrak{g}_0] \not\subseteq \mathfrak{g}_0$). Consider any $x \in \mathfrak{g}_0$ and its Jordan decomposition

$$x = x_{ss} + x_n$$

For any $y \in \mathfrak{h}$, we have $[x, y] = 0$ by definition, hence $[x_{ss}, y] = [x_n, y] = 0$ by (114). Therefore, we have $x_{ss}, x_n \in \mathfrak{g}_0$. If $x_{ss} \notin \mathfrak{h}$, then $\mathfrak{h} \oplus \mathbb{C}x_{ss}$ would be a toral subalgebra, which would contradict the maximality of $\mathfrak{h}$. Therefore, we conclude that $x_{ss} \in \mathfrak{h} \Rightarrow x \in x_n + \mathfrak{h}$, and so

$$\mathrm{ad}_x\Big|_{\mathfrak{g}_0} = \mathrm{ad}_{x_n}\Big|_{\mathfrak{g}_0}$$

is a nilpotent operator on $\mathfrak{g}_0$. By Corollary 4, we conclude that $\mathfrak{g}_0$ is a nilpotent Lie algebra.

Let us first assume that $[\mathfrak{g}_0, \mathfrak{g}_0] \neq 0$. Using Theorem 12, show that nilpotent Lie algebras have the property that their center intersects any non-zero ideal non-trivially. Therefore, there would exist

$$0 \neq z \in \mathfrak{z}(\mathfrak{g}_0) \cap [\mathfrak{g}_0, \mathfrak{g}_0]$$

Since the Killing form is a s.i.b.f., the fact that $z \in [\mathfrak{g}_0, \mathfrak{g}_0]$ implies that

$$(y, z)_\mathfrak{g} = 0, \quad \forall y \in \mathfrak{h}$$

On the other hand, since any $x \in \mathfrak{g}_0 \backslash \mathfrak{h}$ is nilpotent and comutes with $z \in \mathfrak{z}(\mathfrak{g}_0)$, then we have

$$(x, z)_\mathfrak{g} = \mathrm{tr}_\mathfrak{g}(\mathrm{ad}_x \mathrm{ad}_z) = 0 \tag{137}$$

This implies that $(\mathfrak{g}_0, z)_{\mathfrak{g}} = 0$, which contradicts the non-degeneracy of the restricted pairing (134). We therefore conclude that $[\mathfrak{g}_0, \mathfrak{g}_0] = 0$, i.e. $\mathfrak{g}_0$ is abelian. However, we have already seen that any $x \in \mathfrak{g}_0 \backslash \mathfrak{h}$ would have to be nilpotent, so (137) would hold for all $z \in \mathfrak{g}_0$. By the non-degeneracy of the restricted pairing (134), we conclude that $x = 0$, hence $\mathfrak{g}_0 = \mathfrak{h}$.

□

As a consequence of Proposition 16, Cartan subalgebras of complex semisimple Lie algebras $\mathfrak{g}$ exist: just start from the toral subalgebra 0 and enlarge it as much as possible. All Cartan subalgebras of $\mathfrak{g}$ have the same dimension, which is called the **rank** of $\mathfrak{g}$.

9.3

In light of the discussion in the previous Subsection, it makes sense to consider the decomposition (131) for a maximal toral subalgebra, because then the decomposition would be as fine as possible. Thus, we henceforth let $\mathfrak{h}$ be a Cartan subalgebra, and write the decomposition as

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \tag{138}$$

where R simply denotes the set of non-zero linear functionals $\alpha : \mathfrak{h} \to \mathbb{C}$ that have the property that $\mathfrak{g}_\alpha \neq 0$ (and we use the fact that $\mathfrak{g}_0 = \mathfrak{h}$). Since $\mathfrak{g}$ is a finite-dimensional Lie algebra, the set R is finite. It is called the **root system** of $\mathfrak{g}$, and its elements are called **roots**. Because the pairing (134) is non-degenerate when $\alpha + \beta = 0$, we have

$$\alpha \in R \quad \Leftrightarrow \quad -\alpha \in R \tag{139}$$

The $\mathfrak{g}_\alpha$'s that appear in (138) are called the root spaces corresponding the roots α.

Example 5. *Let $\mathfrak{g} = \mathfrak{sl}_n$ and let $\mathfrak{h}$ be the subalgebra of traceless diagonal matrices (with respect to a certain basis). Thus, elements of $\mathfrak{h}$ will be given by*

$$x = (x_1, \dots, x_n) \quad \text{with } x_1 + \dots + x_n = 0$$

We will consider the basis $e_1, \dots, e_n$ of $\mathfrak{h}^*$ given by

$$e_i(x) = x_i$$

With this in mind, the roots are $\{e_i - e_j\}_{1 \le i \neq j \le n}$, with

$$(\mathfrak{sl}_n)_{e_i - e_j} = \mathbb{C}E_{ij}$$

Check the previous claim: all it really says is that if x is the diagonal matrix with entries $(x_1, \dots, x_n)$, then we have $[x, E_{ij}] = (x_i - x_j)E_{ij}$. We conclude that the root decomposition is

$$\mathfrak{sl}_n = \mathfrak{h} \bigoplus_{1 \le i \neq j \le n} \mathbb{C}E_{ij}$$

A number of properties one can observe from Example 5 are quite general, for example the following.

Proposition 17. *The set of roots spans $\mathfrak{h}^*$.*

Proof. If the set of roots failed to span $\mathfrak{h}^*$, then there would be a non-zero element $x \in \mathfrak{h}$ such that $\alpha(x) = 0$ for all $\alpha \in R$. This implies that $[x, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in R$. Adding to this the fact that x commutes with $\mathfrak{h}$ implies that $x \in \mathfrak{z}(\mathfrak{g})$. Since semisimple algebras have trivial center, then $x = 0$.

□

9.4

Another property that one can observe from Example 5 is that all the root spaces are one-dimensional. To prove that this is in fact a general phenomenon, recall the non-degenerate s.i.b.f. $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. Its restriction to $\mathfrak{h}$ is also non-degenerate, and so it provides an isomorphism

$$\boxed{\mathfrak{h} \cong \mathfrak{h}^*} \tag{140}$$

We will write $h_\alpha \in \mathfrak{h}$ for the element corresponding to any root α under the above isomorphism. Moreover, we may define the non-degenerate pairing.

$$(\cdot,\cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$$

via the isomorphism (140).

Lemma 7. *For any root α and any $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, we have*

$$[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha)h_\alpha \tag{141}$$

Proof. For any $h \in \mathfrak{h}$, we have

$$([e_\alpha, f_\alpha], h) = ([h, e_\alpha], f_\alpha) = \alpha(h)(e_\alpha, f_\alpha)$$

By the non-degeneracy of $(\cdot,\cdot)$ restricted to $\mathfrak{h}$, this implies (141).

□

Proposition 18. *For any root α, we have $(\alpha,\alpha) \neq 0$, so we may define*

$$H_\alpha = \frac{2h_\alpha}{(\alpha,\alpha)} \tag{142}$$

If $E_\alpha \in \mathfrak{g}_\alpha$ and $F_\alpha \in \mathfrak{g}_{-\alpha}$ are chosen so that

$$(E_\alpha, F_\alpha) = \frac{2}{(\alpha,\alpha)}$$

then we have the commutation relations

$$[H_\alpha, E_\alpha] = 2E_\alpha, \qquad [H_\alpha, F_\alpha] = -2F_\alpha, \qquad [E_\alpha, F_\alpha] = H_\alpha \tag{143}$$

In other words, $E_\alpha, F_\alpha, H_\alpha$ provide a Lie algebra homomorphism

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$$

*so they are called a $\mathfrak{sl}_2$-**triple**.*

The reason for the normalization (142) is that such a H_α is independent of the choice of s.i.b.f. We will soon see that the $\mathfrak{sl}_2$-triple corresponding to a root α is unique, up to rescaling E_α, F_α by opposite constants.

Proof. of Proposition 18: Let us assume for the purpose of contradiction that $(\alpha, \alpha) = 0$, which would mean that $\alpha(h_\alpha) = 0$. We may choose $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $(e_\alpha, f_\alpha) = 1$ (by the non-degeneracy of the s.i.b.f.) and so Lemma 7 would imply that the subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ generated by $e_\alpha, f_\alpha, h_\alpha$ satisfies the Lie bracket relations

$$[h_\alpha, e_\alpha] = [h_\alpha, f_\alpha] = 0, \quad [e_\alpha, f_\alpha] = h_\alpha$$

This subalgebra is solvable (due to the abelian ideal $\mathbb{C}h_\alpha$) and so Lie's Theorem 11 implies that there is a basis of $\mathfrak{g}$ in which ad_{e_α} and ad_{f_α} are upper triangular. Being the commutator of upper triangular matrices, ad_{h_α} would be strictly upper triangular. However, because $\mathfrak{h}$ is a toral subalgebra, ad_{h_α} is also semisimple. Therefore, we conclude that $\mathrm{ad}_{h_\alpha} = 0$, which implies that $h_\alpha \in \mathfrak{z}(\mathfrak{g})$, which is impossible because semisimple Lie algebras have no center. Having showed that $(\alpha, \alpha) \neq 0$, properties (143) are straightforward computations, which we leave to you.

□

9.5

$\mathfrak{sl}_2$-triples give a powerful tool for the study of semisimple Lie algebras.

Proposition 19. *For any root α, the subspaces $\mathfrak{g}_{\pm\alpha}$ are one-dimensional.*

Proof. Consider any root α, fix a corresponding $\mathfrak{sl}_2$-triple $E_\alpha, F_\alpha, H_\alpha$ and let

$$V_\alpha = \mathbb{C}H_\alpha \bigoplus_{\ell \in \mathbb{Z} \backslash 0} \mathfrak{g}_{\ell\alpha} \tag{144}$$

By (133), the operators ad_{E_α}, ad_{F_α}, ad_{H_α} provide a representation $\mathfrak{sl}_2 \curvearrowright V_\alpha$. The weights of this representation, i.e. the eigenvalues of ad_{H_α}, are equal to the numbers 2ℓ in (144). Thus, V_α is a representation of $\mathfrak{sl}_2$ with all even weights and a one-dimensional 0 weight subspace, so (68) implies that V_α must be irreducible. In particular, this implies that all the root subspaces $\mathfrak{g}_{\pm\alpha}$ are one-dimensional, since they are weight spaces for the aforementioned $\mathfrak{sl}_2$ action. □

Proposition 20. *For any two roots α and β, the number*

$$c_{\alpha\beta} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

is an integer, and $\beta - c_{\alpha\beta}\alpha$ is also a root.

Proof. The numbers in question are equal to the weights of the adjoint action of an $\mathfrak{sl}_2$-triple $E_\alpha, F_\alpha, H_\alpha$ on $\mathfrak{g}_\beta$. Since any finite-dimensional representation of $\mathfrak{sl}_2$ has integer weights, we conclude that $c_{\alpha\beta} \in \mathbb{Z}$. As we have seen in Lecture 5, in any finite-dimensional representation of $\mathfrak{sl}_2$, the operators E^n and F^n provide isomorphisms between the subspaces of weight n and $-n$. In the case at hand, if $0 \neq x \in \mathfrak{g}_\beta$ and $c_{\alpha\beta} < 0$ (respectively $c_{\alpha\beta} > 0$), then $\mathrm{ad}_{E_\alpha}^{-c_{\alpha\beta}}(x) \neq 0$ (respectively $\mathrm{ad}_{F_\alpha}^{c_{\alpha\beta}}(x) \neq 0$). Since the latter elements lie in $\mathfrak{g}_{\beta - c_{\alpha\beta}\alpha}$, we conclude that $\beta - c_{\alpha\beta}\alpha$ is also a root. □

Proposition 21. *The only multiples of a root α which are also roots are α and $-\alpha$.*

Proof. The fact that we have a non-degenerate pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ implies that $-\alpha$ is a root whenever α is a root. On the other hand, if αt were also a root for some $t \in \mathbb{C}\backslash\{\pm 1\}$, then Proposition 20 would imply that $2t \in \mathbb{Z}$. However, switching the roles of α and αt would also imply that $2t^{-1} \in \mathbb{Z}$, which only leaves the possibility that $t \in \{\pm\frac{1}{2}, \pm 2\}$. Let us assume without loss of generality that $t = \pm 2$. Then as we saw in Proposition 19, V_α must be an irreducible representation with respect to the $\mathfrak{sl}_2$ triple $E_\alpha, F_\alpha, H_\alpha$. This would imply that $\mathfrak{g}_{2\alpha} \subseteq \mathrm{ad}_{E_\alpha}(\mathfrak{g}_\alpha)$, which is impossible since we already showed that $\mathfrak{g}_\alpha$ is the one-dimensional space $\mathbb{C}E_\alpha$. $\square$

Lecture 10

10.1

Propositions 17, 19, 20, 21 can be unified in the following abstract definition of a root system (which we will shortly see provides a model for all complex semisimple Lie algebras).

Definition 26. *An* ***abstract root system*** *is a $\mathbb{R}$-vector space U endowed with an inner product*

$$U \times U \xrightarrow{(\cdot,\cdot)} \mathbb{R} \tag{145}$$

together with a finite set $R \subset U \backslash 0$ such that

$$R \text{ spans } U \tag{146}$$

$$\text{if } \alpha \in R \text{ then } k\alpha \begin{cases} \in R & \text{if } k \in \{-1, 1\} \\ \notin R & \text{if } k \in \mathbb{R}\backslash\{-1, 1\} \end{cases} \tag{147}$$

$$\text{if } \alpha, \beta \in R \text{ then } c_{\alpha\beta} := \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z} \tag{148}$$

$$\text{if } \alpha, \beta \in R \text{ then } s_\alpha(\beta) = \beta - c_{\alpha\beta}\alpha \in R \tag{149}$$

The **rank** of a root system is defined to be the dimension of U.

10.2

Axioms (148) and (149) might seem contrived at first, but they have a geometric meaning in terms of reflections. The former of these axioms is a statement about the angle between the vectors α and β. Meanwhile, the latter axiom concerns the function

$$s_\alpha : U \to U, \qquad s_\alpha(\lambda) = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha \tag{150}$$

which is none other than the reflection across the hyperplane $\alpha^\perp$ perpendicular to α: thus (149) merely says that the root system is preserved by such reflections.

Definition 27. *The* ***(abstract) Weyl group*** *is the subgroup of $GL(U)$ (actually of the orthogonal group, because reflections preserve the inner product* (145)*) generated by the reflections $\{s_\alpha\}_{\alpha \in R}$.*

Note that the Weyl group is always finite, because any element of $GL(U)$ which fixes every root must be the identity due to (146). The reason for the word "abstract" in Definition 27 is to differentiate it from the Weyl group of a complex semisimple Lie algebra $\mathfrak{g}$, which is defined as

$$W = N_G(H)/H \tag{151}$$

where G is the simply connected complex Lie group with Lie algebra $\mathfrak{g}$ (see Theorem 9), and H is a closed Lie subgroup with Lie algebra given by a Cartan subalgebra of G (see Theorem 7; such a subgroup H is called a maximal torus). As you can expect, W in (151) is isomorphic to the abstract Weyl group corresponding to the root system of $\mathfrak{g}$, though we will not prove it.

Example 6. *For the root system of $\mathfrak{sl}_n$, we have that*

$$s_{e_i - e_j}(\dots, x_i, \dots, x_k, \dots, x_j, \dots) = (\dots, x_j, \dots, x_k, \dots, x_i, \dots)$$

The corresponding Weyl group is easily seen to be the symmetric group S_n.

10.3

It turns out that the axioms of a root system are very restrictive: in rank 1, this is probably not so surprising, since the only root system in $\mathbb{R}$ is the root system associated to the Lie algebra $\mathfrak{sl}_2$: this root system is called "type A_1", where the subscript denotes the rank.

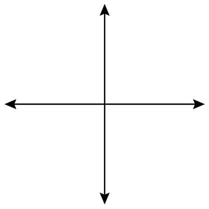
Things are a bit more interesting in rank 2, i.e. $U = \mathbb{R}^2$. In this case, the angle θ between two non-collinear roots $\alpha, \beta \in R$ is given by the following formula (we let $|\alpha| = \sqrt{(\alpha, \alpha)}$)

$$\cos\theta = \frac{(\alpha,\beta)}{|\alpha||\beta|} = \frac{c_{\alpha\beta}}{2} \cdot \frac{|\alpha|}{|\beta|} = \frac{c_{\beta\alpha}}{2} \cdot \frac{|\beta|}{|\alpha|} \quad \Rightarrow \quad (\cos\theta)^2 = \frac{c_{\alpha\beta}c_{\beta\alpha}}{4}$$

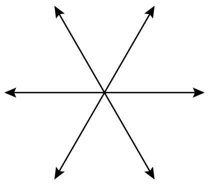
Therefore, we must have $0 \leq c_{\alpha\beta}c_{\beta\alpha} < 4$. As axiom (148) requires that $c_{\alpha\beta}$ and $c_{\beta\alpha}$ be integers, we only have the following options (we assume without loss of generality that $|\alpha| \leq |\beta|$, so $|c_{\alpha\beta}| \geq |c_{\beta\alpha}|$):

- $c_{\alpha\beta} = 0 \Leftrightarrow c_{\beta\alpha} = 0$, which implies $\theta = \frac{\pi}{2}$
- $c_{\alpha\beta} = c_{\beta\alpha} = 1$, which implies $|\alpha| = |\beta|$ and $\theta = \frac{\pi}{3}$
- $c_{\alpha\beta} = c_{\beta\alpha} = -1$, which implies $|\alpha| = |\beta|$ and $\theta = \frac{2\pi}{3}$
- $c_{\alpha\beta} = 2$ and $c_{\beta\alpha} = 1$, which implies $|\alpha|\sqrt{2} = |\beta|$ and $\theta = \frac{\pi}{4}$
- $c_{\alpha\beta} = -2$ and $c_{\beta\alpha} = -1$, which implies $|\alpha|\sqrt{2} = |\beta|$ and $\theta = \frac{3\pi}{4}$
- $c_{\alpha\beta} = 3$ and $c_{\beta\alpha} = 1$, which implies $|\alpha|\sqrt{3} = |\beta|$ and $\theta = \frac{\pi}{6}$
- $c_{\alpha\beta} = -3$ and $c_{\beta\alpha} = -1$, which implies $|\alpha|\sqrt{3} = |\beta|$ and $\theta = \frac{5\pi}{6}$

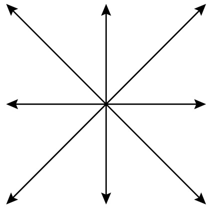
With this in mind, the following are easily seen to be root systems, because the angle between any two roots is admissible by the discussion above (the notation $A_1 \times A_1$, A_2, $B_2 \cong C_2$ and G_2 will be explained in Theorem 19).



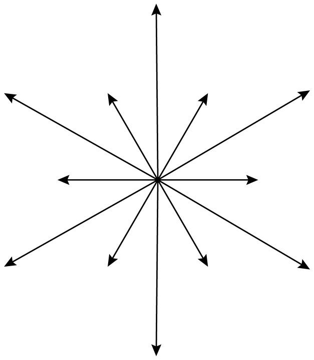
$A_1 \times A_1$



A_2



$B_2 = C_2$



G_2

What is more interesting is that the above are all the rank 2 root systems, up to linear transformations. Indeed, consider any rank 2 root system R and look at the most obtuse angle between two non-collinear roots: if this angle is $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, $\frac{5\pi}{6}$, the root system is $A_1 \times A_1$, A_2, $B_2 = C_2$, G_2, respectively (prove this yourselves; the idea is that once you draw two vectors with the most obtuse angle between them, the fact that R is preserved under the reflections (150) means that R must contain a copy R' of the root system of type $A_1 \times A_1$, A_2, $B_2 = C_2$, G_2, respectively; but if R contained any other root α, you could find some root $\beta \in R'$ which would violate the angle and length conditions in the bullets above).

10.4

What about root systems of arbitrary rank? The bulleted list in the previous Subsection actually pertains to any pair of non-collinear elements of any root system, and thus controls the angle between any pair of roots. For instance, we ask you to prove the following Lemma by examining all the rank 2 systems above.

Lemma 8. *If $\alpha \neq \beta \in R$ have the property that $(\alpha, \beta) > 0$, then $\alpha - \beta \in R$.*

Let us now consider arbitrary root systems R, and develop some further tools to classify them.

Definition 28. *Any hyperplane in U that does not intersect R determines a decomposition*

$$R = R^+ \sqcup R^- \tag{152}$$

*into **positive** and **negative** roots, depending on which side of U they lie. Clearly, $R^- = -R^+$. A **simple** root is a positive root which cannot be written as a sum of two or more positive roots.*

For the root system associated to $\mathfrak{sl}_n$, the usual choice is to let

$$R^+ = \left\{ e_i - e_j \middle| 1 \leq i < j \leq n \right\}$$
$$R^- = \left\{ e_i - e_j \middle| 1 \leq j < i \leq n \right\}$$

The simple roots are then $\alpha_i = e_i - e_{i+1}$, with $i \in \{1, \dots, n-1\}$.

Proposition 22. *(a) Every positive root can be written uniquely as a sum of simple roots.*

(b) The simple roots determine a basis of U (so there are as many of them as the rank of R).

Proof. (a) We may successively decompose any positive root α into sums of positive roots. This process must terminate after finitely many steps (since there are finitely many positive roots, and all of them are at least a fixed distance away from the hyperplane separating R^+ from R^-) and when it terminates, we will have written α as a sum of simple roots.

(b) By the previous part and axiom (146), the simple roots span U. To prove that they are linearly independent, note that

$$\alpha \neq \beta \text{ simple} \quad \Rightarrow \quad (\alpha, \beta) \leq 0 \tag{153}$$

(indeed, otherwise Lemma 8 would imply that either $\alpha - \beta$ or $\beta - \alpha$ is a positive root, which would contradict the simplicity of α and β). However, it is a classic and easy exercise to show that any set of vectors which have all non-acute angles between them must be linearly independent. □

10.5

The following result, which will occupy the remainder of this lecture, shows that a set of simple roots determines the entire R. In what follows, we fix a set of simple roots $\alpha_1, \ldots, \alpha_r$ of R.

Theorem 18. *The Weyl group W is generated by the* ***simple reflections***

$$\boxed{\Big\{s_i = s_{\alpha_i}\Big\}_{i \in \{1,\ldots,r\}}} \tag{154}$$

and any root can be obtained by some element of W acting on some simple root α_i.

The theorem above says that a Weyl group is a particular case of a so-called Coxeter group. The first step in proving Theorem 18 is to systematize the freedom we had in choosing the decomposition (152).

Definition 29. *Consider the hyperplanes $\alpha^\perp$ perpendicular to the the roots $\alpha \in R$. A connected component $\mathcal{C}$ of*

$$U \Big\backslash \bigcup_{\alpha \in R} \alpha^\perp$$

is called a ***Weyl chamber****. The boundary hyperplanes of a chamber are often called its walls.*

By definition, a Weyl chamber is the set of all $x \in U$ such that (x, α) has a given sign for all roots $\alpha \in R$. As the decomposition (152) partitions the roots into two sets, depending on whether their pairing with a given $x \in U$ is positive or negative, we conclude that the decomposition itself only depends on the Weyl chamber of x. In other words, a choice of positive/negative roots is equivalent to a choice of positive/negative Weyl chamber $\mathcal{C}^\pm$ (namely the chamber consisting of x's whose inner product with the positive/negative roots is > 0).

Recall that the linear transformations s_α of (150) are by definition given by reflecting in the hyperplanes $\alpha^\perp$. Since (149) says that the W action takes any root β to a root β', it follows that W takes the hyperplane $\beta^\perp$ to $\beta'^\perp$ and thus W takes Weyl chambers to Weyl chambers.

Proposition 23. *The Weyl group action on the Weyl chambers is transitive.*

Proof. It is a common feature of hyperplane arrangements that any two chambers $\mathcal{C}$ and $\mathcal{C}'$ can be connected by a sequence of Weyl chambers

$$\mathcal{C} = \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \mathcal{C}_k = \mathcal{C}'$$

such that $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ are adjacent, i.e. separated by a single hyperplane $\alpha^\perp$. That means that the reflection s_α takes $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$, which implies that some element of the Weyl group takes $\mathcal{C}$ to $\mathcal{C}'$. $\square$

Corollary 7. *For any two sets of positive roots $R = R^+ \sqcup R^- = R'^+ \sqcup R'^-$, there exists an element of the Weyl group taking $R^\pm$ to $R'^\pm$.*

Proof. Let $\mathcal{C}$ and $\mathcal{C}'$ be the positive Weyl chambers with respect to R^+ and R'^+, respectively. Proposition 23 says that there exists an element w of the Weyl group which sends $\mathcal{C}$ to $\mathcal{C}'$. Since elements of the Weyl group act by orthogonal matrices, they preserve the scalar product, so a positive root with respect to $\mathcal{C}$ will be sent to a positive root with respect to $\mathcal{C}'$ (check this fact). Since the number of positive roots is always $\frac{|R|}{2}$, this proves that $w(R^+) = R'^+$. □

It is also easy to see that a Weyl group element taking R^+ to R'^+ must also take the simple roots inside R^+ to the simple roots inside R'^+.

10.6

Let us now fix a decomposition (152). By definition, the positive Weyl chamber $\mathcal{C}^+$ is defined by the property $(x, \alpha) > 0$ for all $\alpha \in R^+$. By Proposition 22, this is equivalent to

$$(x, \alpha_i) > 0$$

where $\alpha_1, \dots, \alpha_r$ are the simple roots inside R^+. This implies that the walls of the chamber $\mathcal{C}^+$ are actually $\alpha_1^\perp, \dots, \alpha_r^\perp$, since it is impossible to encounter any other wall $\alpha^\perp$ (i.e. have $(x, \alpha) = 0$) without first encountering one of the walls $\alpha_1^\perp, \dots, \alpha_r^\perp$ (i.e. have $(x, \alpha_i) = 0$ for some $i \in \{1, \dots, r\}$).

Proposition 24. *For any chamber $\mathcal{C}$, there exist simple reflections $i_1, \dots, i_k$ such that*

$$\mathcal{C} = s_{i_1} \dots s_{i_k}(\mathcal{C}^+) \tag{155}$$

Proof. By Proposition 23, there exists a sequence of Weyl chambers

$$\mathcal{C}^+ = \mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{k-1}, \mathcal{C}_k = \mathcal{C}$$

such that $\mathcal{C}_{\ell-1}$ is separated from $\mathcal{C}_\ell$ by a single hyperplane. We will prove that each $\mathcal{C}_\ell$ can be written in the form (155) by induction on ℓ. So assume that we have

$$\mathcal{C}_{\ell-1} = s_{i_1} \dots s_{i_{\ell-1}}(\mathcal{C}^+) \tag{156}$$

and $\mathcal{C}_\ell$ is separated from $\mathcal{C}_{\ell-1}$ by the hyperplane $\alpha^\perp$. This hyperplane must be of the form $s_{i_1} \dots s_{i_{\ell-1}}(\alpha_{i_\ell}^\perp)$ for some $i_\ell \in I$, because the walls of $\mathcal{C}^+$ are the $\alpha_i^\perp$'s. We conclude that

$$\alpha^\perp = s_{i_1} \dots s_{i_{\ell-1}}(\alpha_{i_\ell}^\perp) \quad \Leftrightarrow \quad \alpha = s_{i_1} \dots s_{i_{\ell-1}}(\alpha_{i_\ell})$$

The obvious formula

$$\boxed{s_{w(\alpha)} = w s_\alpha w^{-1}} \tag{157}$$

$\forall w \in W, \alpha \in R$, then allows us to deduce the required identity $\mathcal{C}_\ell = s_{i_1} \dots s_{i_\ell}(\mathcal{C}^+)$ from (156). □

Proof. of Theorem 18: Any root $\alpha \in R$ is simple with respect to some decomposition $R = R'^+ \sqcup R'^-$ (just choose the hyperplane separating R'^+ and R'^- very close to α). Proposition 24 implies that there exists a product of simple reflections

$$w = s_{i_1} \dots s_{i_k}$$

which takes the fixed decomposition $R^\pm$ to our $R'^\pm$, and as such takes some simple root α_i to α: $w(\alpha_i) = \alpha$. By formula (157), s_α is therefore a product of simple reflections. Since the s_α's generate W, we conclude that W is generated by simple reflections.

□

Lecture 11

11.1

In the previous lecture, we showed that one can reconstruct a root system from a set of simple roots $\{\alpha_1, \dots, \alpha_r\}$. In turn, we will soon see that such a set of simple roots is completely determined (up to an angle-preserving linear transformation of U) by the following notion that encodes the lengths of simple roots and the angles between them.

Definition 30. *A **Cartan matrix** of rank r is a square matrix with integer entries*

$$C = (c_{ij})_{1 \le i,j \le r} \tag{158}$$

such that

- $c_{ii} = 2$ *and* $c_{ij} \le 0$ *for all* $i \neq j$.
- $c_{ij} = 0 \Leftrightarrow c_{ji} = 0$.
- $C = DS$, *where D is a diagonal matrix with positive entries on the diagonal and S is a positive-definite symmetric matrix.*

We can associate a Cartan matrix to any root system R, by letting $c_{ij} = c_{\alpha_i \alpha_j}$ for some set of simple roots $\alpha_1, \dots, \alpha_r$. One chooses the matrices D and S to have entries $\frac{2}{(\alpha_i, \alpha_i)}$ and (α_i, α_j), respectively, with the notation in Definition 26. Then the positive-definiteness of S is equivalent to the fact that $\alpha_1, \dots, \alpha_r$ are a basis of E. For instance, the Cartan matrix of $\mathfrak{sl}_n$ is

$$\begin{pmatrix} 2 & -1 & 0 & \vdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \vdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \vdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \vdots & 0 & -1 & 2 \end{pmatrix} \tag{159}$$

Note that positive-definiteness implies that in a Cartan matrix we have the equation

$$0 \le c_{ij} c_{ji} < 4$$

just like we saw in Subsection 10.3. We typically refer to <u>the</u> Cartan matrix of a root system because changing the choice of simple roots merely has the effect of permuting the rows and columns of C.

Proposition 25. *Up to an angle-preserving linear transformation of U, a root system R is completely determined by its Cartan matrix.*

Proof. The key insight is that given a Cartan matrix C, the decomposition

$$C = DS$$

is unique up to rescaling D and S by inverse amounts (this is because multiplying D on the right by a non-scalar diagonal matrix D' would have to be counterbalanced by multiplying S on the left by D'^{-1}, but this would spoil the symmetry of S). Therefore, once the Cartan matrix of a root system is given, the inner products (α_i, α_j) are all determined up to constant multiple. It is easy to show that this determines the simple roots $\alpha_1, \dots, \alpha_r$ up to an angle-preserving linear transformation. But once a collection of simple roots is fixed, Theorem 18 implies that any root can be obtained from them by successively applying reflections $s_i = s_{\alpha_i}$. $\square$

11.2

Beside Cartan matrices, which provide a numerical characterization of simple roots in a root system, we also have the following graphical realization of the same information.

Definition 31. *The* ***Dynkin diagram*** *associated to a root system R is the graph with vertex set $\{1, \dots, r\}$, and*

- *0 edges between i and j if the angle between α_i and α_j is $\frac{\pi}{2}$*
- *1 edge between i and j if the angle between α_i and α_j is $\frac{2\pi}{3}$*
- *2 edges between i and j if the angle between α_i and α_j is $\frac{3\pi}{4}$*
- *3 edges between i and j if the angle between α_i and α_j is $\frac{5\pi}{6}$*

(by the discussion in Subsections 10.3 and 10.4, the above are the only possibilities for angles between simple roots). If there are multiple edges between two vertices, we draw an arrow from the one corresponding to a longer root to the one corresponding to a shorter root.

It is easy to see that there is a one-to-one correspondence

$$\Big(\text{Cartan matrices}\Big) \leftrightarrow \Big(\text{Dynkin diagrams}\Big) \tag{160}$$

wherein the set of rows/columns of a Cartan matrix is identified with the vertex set $\{1, \dots, r\}$ of a Dynkin diagram. For any two vertices $i \neq j \in \{1, \dots, r\}$, the number of edges between i and j in the Dynkin diagram are perfectly encoded in the non-positive integer entries c_{ij} and c_{ji} of the Cartan matrix, as explained in the bulleted list of Subsection 10.3. More explicitly, since filling out a Cartan matrix and drawing a Dynkin diagram are rank 2 tasks (i.e. ones which you perform by considering any principal 2×2 submatrix and any 2-vertex subgraph at a time) then the correspondence (160) is completely determined by the assignment

$$\begin{array}{ccc}
\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \leftrightarrow & A_1 \times A_1 \\
\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} & \leftrightarrow & A_2 \\
\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} & \leftrightarrow & B_2 = C_2 \\
\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} & \leftrightarrow & G_2
\end{array}$$

(see below for the notation $A_n, B_n, C_n, D_n, E_{6,7,8}, F_4, G_2$ of Dynkin diagrams).

Definition 32. *Given root systems $R \subset U$ and $R' \subset U'$, their* ***direct sum*** *is the root system*

$$(R, 0) \sqcup (0, R') \subset E \oplus E'$$

A root system which is not isomorphic to a non-trivial direct sum is called ***irreducible****.*

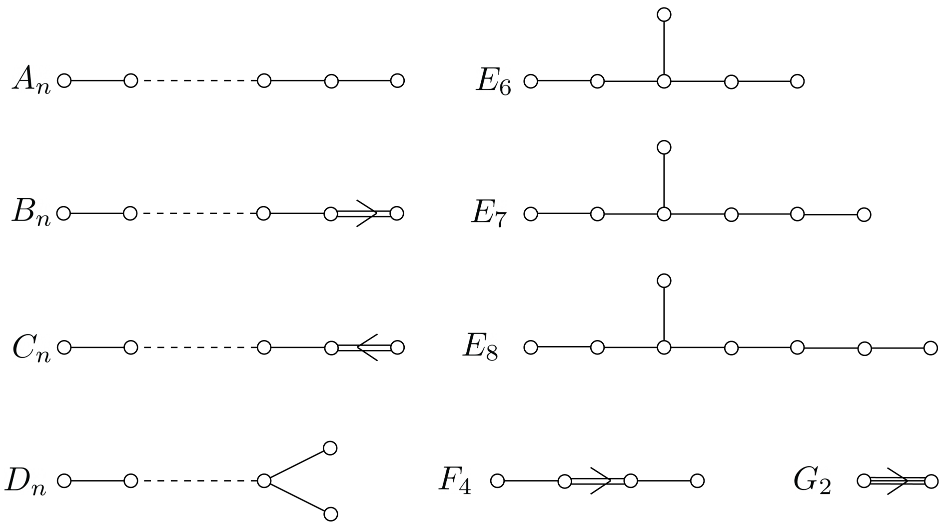
Proposition 26. *In an irreducible root system R, either all roots have the same length (in which case R is called* ***simply laced****) or there are only two possible values for the root lengths (in which case they are called* ***short*** *and* ***long*** *roots, the latter being $\sqrt{2}$ or $\sqrt{3}$ times longer than the fomer).*

Proof. In an irreducible root system, the action $W \curvearrowright U$ is irreducible (hence the terminology), because the orthogonal complement of any W-invariant subspace with respect to the inner product (145) would also be W-invariant. As a consequence, the W-orbit of any root α spans U, so for any other root β there must exist $w \in W$ such that $(w(\alpha), \beta) \neq 0$. The bulleted list in Subsection 10.3 then implies that the length of β and the length of α (which is equal to the length of $w(\alpha)$ for any $w \in W$) must differ by a ratio of $1, \sqrt{2}, \sqrt{3}$. Thus, if the roots could have three or more lengths, we could always find a pair of them which differ by a ratio other than $1, \sqrt{2}, \sqrt{3}$, thus contradicting the previous sentence. ☐

11.3

It is easy to see that a Cartan matrix (respectively Dynkin diagram) corresponds to an irreducible root system if and only if it is not block diagonal (respectively connected). Therefore, the task is to classify irreducible Dynkin diagrams.

Theorem 19. *Any irreducible Dynkin diagram is one of the following list*

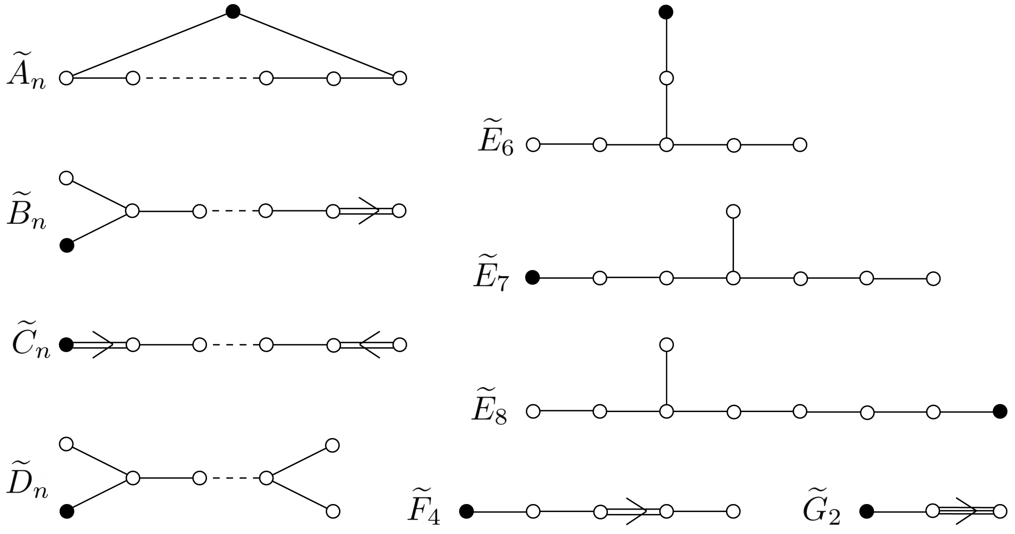


where the index denotes the number of vertices in the diagram.

Proof. The fact that the pictures above represent Dynkin diagrams comes from the fact that the corresponding Cartan matrices $C = DS$ have positive determinant

$$\begin{aligned}
\det A_n &= n+1 \\
\det B_n &= 2 \\
\det C_n &= 2 \\
\det D_n &= 4 \\
\det E_{6,7,8} &= 3,2,1 \\
\det F_4 &= 1 \\
\det G_2 &= 1
\end{aligned}$$

and that the top left corners of the corresponding S matrices also have positive determinant (since they are also Cartan matrices of the Dynkin diagrams listed above). By Sylvester's criterion, these matrices S are positive definite. On the other hand, the following pictures are <u>not</u> Dynkin diagrams, as the corresponding matrices S have determinant 0 [5].



These are called **extended Dynkin diagrams**, and they reflect the representation theory of affine Lie algebras (these are infinite-dimensional Lie algebras obtained from $\mathfrak{g}[t^{\pm 1}]$ where $\mathfrak{g}$ is a complex semisimple Lie algebra, that we will not study in the present course). But the relevance of the pictures above to us is that no Dynkin diagram can contain an extended Dynkin diagram as a subdiagram, or else its symmetrized Cartan matrix S would have a principal minor of determinant 0 (and thus fail to be positive definite). Thus, we have the following observations:

- A Dynkin diagram cannot contain a cycle: indeed, if we had a cycle $i_1, i_2, \dots, i_k, i_{k+1} = i_1$ with $k \geq 3$, then we would be able to contradict the positive-definiteness of $S = (d_{ij})_{1 \leq i,j \leq r}$ as follows

$$\sum_{s=1}^{k} (d_{i_s i_s} + 2 d_{i_s i_{s+1}}) \leq 0$$

---

[5] The convention for $\widetilde{A}_1$ is that its Cartan matrix is $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

whch holds because $C = DS$ implies that $c_{ij} = \frac{2d_{ij}}{d_{ii}}$ for all $i \neq j \in \{1, \dots, r\}$, and we have

$$d_{ii} + d_{jj} \leq -4d_{ij} \Leftrightarrow \frac{1}{-c_{ij}} + \frac{1}{-c_{ji}} \leq 2$$

for all negative integers c_{ij}, c_{ji}.

- If a Dynkin diagram contains a triple edge, then it is just G_2 (otherwise it would contain a copy of the extended Dynkin diagram $\widetilde{G}_2$, or one whose matrix S is negative-definite)

- if a Dynkin diagram contains a double edge, then it is just B_n, C_n or F_4, because otherwise it would contain a copy of $\widetilde{A}_1, \widetilde{B}_n$, $\widetilde{C}_n$ or $\widetilde{F}_4$.

- if a Dynkin diagram has only single edges and no cycles, then it is either one of $A_n, D_n, E_{6,7,8}$, or else it would contain a copy of $\widetilde{D}_n$ or $\widetilde{E}_{6,7,8}$.

□

11.4

We summarize the discussion above by claiming the existence of one-to-one correspondences

$$\Big\{\text{semisimple complex Lie algebras}\Big\} \leftrightarrow \Big\{\text{root systems}\Big\} \tag{161}$$

and

$$\Big\{\text{root systems}\Big\} \leftrightarrow \Big\{\text{Cartan matrices} \overset{(160)}{\leftrightarrow} \text{Dynkin diagrams}\Big\} \tag{162}$$

with the arrows $\rightarrow$ in the above equations being provided in Lectures 9 and 11, respectively. The backwards arrows will be provided by the construction in Lecture 12. Moreover, the one-to-one correspondences (160), (161), (162) match the following particular types of objects

- simple complex Lie algebras,

- irreducible root systems,

- non-block diagonal Cartan matrices / connected Dynkin diagrams.

Indeed, we have already seen that A_n is the Dynkin diagram of the Lie algebra $\mathfrak{sl}_{n+1}$. We also have

$$\begin{aligned} &B_n \text{ is the Dynkin diagram of } \mathfrak{o}_{2n+1} \\ &C_n \text{ is the Dynkin diagram of } \mathfrak{sp}_{2n} \\ &D_n \text{ is the Dynkin diagram of } \mathfrak{o}_{2n} \text{ for } n > 1 \end{aligned}$$

As for the simple Lie algebras that correspond to types E, F, G, we will construct them abstractly in the next lecture. If we take a semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}$, the Dynkin diagram associated to $\mathfrak{g}$ will be the disconnected union of the Dynkin diagrams associated to $\mathfrak{g}_1, \dots, \mathfrak{g}_k$.

Lecture 12

12.1

In the previous lectures, we showed how to perform the following operations

$$\text{semisimple Lie algebras} \rightsquigarrow \text{root systems} \rightsquigarrow \text{Cartan matrices} \leftrightarrow \text{Dynkin diagrams}$$

We will now show how to reconstruct a semisimple Lie algebra from the Cartan matrix / Dynkin diagram of the corresponding root system. Before we do so, we must be able to define Lie algebras by generators and relations. The following discussion is completely analogous to that of groups, that you encountered in Math 211. Let $\mathbb{K}$ be any field of characteristic 0.

Definition 33. *Let S be any set called an **alphabet**. The free associative algebra on S is*

$$A_S = \bigoplus_{s_1 \dots s_k \text{ word in } S} \mathbb{K} s_1 \dots s_k$$

*with the operation given by concatenation of words. We can think of A_S as a Lie algebra with respect to commutator, and we define the **free Lie algebra** on S*

$$\boxed{\mathfrak{f}_S} \tag{163}$$

as the Lie subalgebra of A_S generated by all one-letter words.

If you like universal properties, the free Lie algebra is determined up to isomorphism by the fact that for any Lie algebra $\mathfrak{g}$, a choice of elements $\{x_s \in \mathfrak{g}\}_{s \in S}$ extends uniquely to a Lie algebra homomorphism $\mathfrak{f}_S \to \mathfrak{g}$. But perhaps more explicitly, you should think of $\mathfrak{f}_S$ as consisting of all $\mathbb{K}$-linear combination of symbols

$$[\dots[[s_1, s_2], s_3], [s_4, s_5] \dots] \tag{164}$$

(for any $s_1, s_2, \dots \in S$ and any distribution of square brackets) that are related by antisymmetry and the Jacobi identity. While this may seem complicated, it is controlled by beautiful combinatorics. For instance, let us fix a total order on the set S, which determines a lexicographic order on the set of all words written with the alphabet S. We call a word w **Lyndon** (also known as **Shirshov**) if it is lexicographically smaller than all of its proper suffixes. Then a classic result is that

$$\mathfrak{f}_S = \bigoplus_{w \text{ Lyndon word}} \mathbb{K} x_w$$

where $x_w \in \mathfrak{f}_S$ are defined recursively by $x_s = s$ for any $s \in S$, while for any Lyndon word w of length ≥ 2 we set

$$x_w = [x_{w'}, x_{w''}]$$

where w'' is the longest suffix of $w = w'w''$ which is also a Lyndon word (with this choice, it is not hard to show that the prefix w' is also a Lyndon word).

Definition 34. *Let R denote any set of **relations**, i.e. $\mathbb{K}$-linear combinations of symbols* (164). *Then*

$$\boxed{\mathfrak{f}_{S|R} = \mathfrak{f}_S \Big/ \Big(\textit{ideal generated by } R\Big)} \tag{165}$$

is called the Lie algebra generated by S modulo relations R.

For example, if R is the set of $\{[s, s']\}_{s, s' \in S}$, then (165) is called the free abelian Lie algebra on S, and it is simply isomorphic to $\oplus_{s \in S} \mathbb{K} s$ with trivial Lie bracket.

12.2

We henceforth work over the ground field $\mathbb{C}$. For any Cartan matrix $C = (c_{ij})_{i,j \in \{1,\dots,r\}}$, we define

$$\boxed{\mathfrak{g}_C} \tag{166}$$

to be the Lie algebra generated by symbols $\{E_i, F_i, H_i\}_{i \in \{1,\dots,r\}}$ modulo the relations

$$[H_i, H_j] = 0 \tag{167}$$

$$[H_i, E_j] = c_{ij} E_j \tag{168}$$

$$[H_i, F_j] = -c_{ij} F_j \tag{169}$$

$$[E_i, F_j] = \delta_{ij} H_i \tag{170}$$

for all $i, j \in \{1, \dots, r\}$, as well as

$$\mathrm{ad}_{E_i}^{1-c_{ij}}(E_j) = 0 \tag{171}$$

$$\mathrm{ad}_{F_i}^{1-c_{ij}}(F_j) = 0 \tag{172}$$

for all distinct $i, j \in \{1, \dots, r\}$. The main result of this Lecture is the following theorem of Serre.

Theorem 20. *For any irreducible Cartan matrix C, the Lie algebra $\mathfrak{g}_C$ is finite-dimensional and simple. Its root system has associated Cartan matrix precisely equal to C.*

Moreover, it is easy to see that if $C = C_1 \oplus C_2$, then $\mathfrak{g}_C \cong \mathfrak{g}_{C_1} \oplus \mathfrak{g}_{C_2}$. Coupling this with Lemma 5 allows us to extend Theorem 20 to arbitrary Cartan matrices, by replacing the word "simple" with "semisimple".

Example 7. *When C is the Cartan matrix (159) of type A_{n-1}, the isomorphism $\mathfrak{g}_C \cong \mathfrak{sl}_n$ is given by*

$$E_i \rightsquigarrow E_{i,i+1}, \quad F_i \rightsquigarrow E_{i+1,i}, \quad H_i \rightsquigarrow E_{ii} - E_{i+1,i+1}$$

It is easy to check by hand that relations (167)-(172) hold in $\mathfrak{sl}_n$, which gives a homomorphism $\mathfrak{g}_C \to \mathfrak{sl}_n$. It is also easy to check that this homomorphism is surjective (because the matrices $E_{i,i+1}$ and $E_{i+1,i}$ generate $\mathfrak{sl}_n$ as a Lie algebra) and so it must be an isomorphism due to $\mathfrak{g}_C$ being simple.

12.3

We start by identifying relations (167)-(172) in any complex semisimple Lie algebra $\mathfrak{g}$. Fix a s.i.b.f. $(\cdot, \cdot)$ and a set of simple roots $\alpha_1, \dots, \alpha_r$ of $\mathfrak{g}$. As in Proposition 18, we can pick elements

$$E_{\alpha_i} \in \mathfrak{g}_{\alpha_i} \quad \text{and} \quad F_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$$

such that $[E_{\alpha_i}, F_{\alpha_i}] = H_{\alpha_i}$ determine an $\mathfrak{sl}_2$-triple, and so satisfy relations (167)-(170) with E_i replaced by E_{α_i} etc. To prove that these elements also satisfy (171), consider any $i \neq j$ and make

$$\bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}_{\alpha_j + \ell \alpha_i}$$

into a representation of $\mathfrak{sl}_2$ via the operators $\mathrm{ad}_{E_{\alpha_i}}, \mathrm{ad}_{F_{\alpha_i}}, \mathrm{ad}_{H_{\alpha_i}}$. As we have seen in the proof of Propositions 19 and 20, the weight of the ℓ-th direct summand in the formula above is

$$\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} + 2\ell = c_{ij} + 2\ell$$

(check this fact). Since $\mathrm{ad}_{F_{\alpha_i}}(E_{\alpha_j}) = 0$ by (170), then we conclude that E_{α_j} is a vector of lowest weight c_{ij}. By Corollary 1, this implies that

$$\mathrm{ad}_{E_{\alpha_i}}^{1-c_{ij}}(E_{\alpha_j}) = 0$$

which is precisely (171). Relation (172) is proved analogously. Therefore, we conclude that the assignments $E_i \mapsto E_{\alpha_i}$, $F_i \mapsto F_{\alpha_i}$, $H_i \mapsto H_{\alpha_i}, \forall i \in \{1, \dots, r\}$ determine a Lie algebra homomorphism

$$\mathfrak{g}_C \to \mathfrak{g} \tag{173}$$

where C is the Cartan matrix associated to $\mathfrak{g}$.

Proposition 27. *The E_{α_i} and F_{α_i} defined above generate any complex semisimple Lie algebra $\mathfrak{g}$, i.e. the homomorphism* (173) *is surjective.*

Proof. Since any root is a non-negative integer combination of positive roots, it suffices to prove the following statement in any complex semisimple Lie algebra $\mathfrak{g}$: if α and β are roots such that $\alpha + \beta$ is also a root, then

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta} \tag{174}$$

(the inclusion $\subseteq$ between the sets above is quite general, see (133)). To see this, one picks an $\mathfrak{sl}_2$ triple $E_\alpha, F_\alpha, H_\alpha$ for the positive root α, and uses it to construct a representation of $\mathfrak{sl}_2$ on

$$\bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}_{\beta+\ell\alpha}$$

In any representation of $\mathfrak{sl}_2$, higher weight subspaces are obtained from the action of E on lower weight subspaces. In the case at hand, since $\mathfrak{g}_{\alpha+\beta}$ has higher weight than $\mathfrak{g}_\beta$, then we must have

$$\mathfrak{g}_{\alpha+\beta} = \mathrm{ad}_{E_\alpha}(\mathfrak{g}_\beta)$$

which precisely implies (174). □

12.4

Having showed that the homomorphism (173) is surjective, it will follow from Theorem 20 that it is an isomorphism: thus, there exists a unique simple complex Lie algebra with any given irreducible root system. As a stepping stone to proving Theorem 20, let us understand the Lie algebra

$$\widetilde{\mathfrak{g}}_C$$

freely generated by $\{E_i, F_i, H_i\}_{1 \leq i \leq r}$ modulo relations (167)-(170), as per Definition 34. This Lie algebra is graded by the **root lattice**

$$Q = \left\{ n_1\alpha_1 + \dots + n_r\alpha_r \middle| n_1, \dots, n_r \in \mathbb{Z} \right\} \tag{175}$$

via

$$\deg E_i = \alpha_i, \quad \deg F_i = -\alpha_i, \quad \deg H_i = 0 \tag{176}$$

(a Lie algebra is called graded if $\deg[x, y] = \deg x + \deg y$, which you can prove to be the case for $\widetilde{\mathfrak{g}}_C$). We let Q^+ (resp. Q^-) be the non-negative integer span of the positive (resp. negative) roots.

Proposition 28. *Consider the following subalgebras of* $\widetilde{\mathfrak{g}}_C$

$\widetilde{\mathfrak{n}}_C^+$ *spanned by arbitrary iterated Lie brackets of* E_i*'s*

$\widetilde{\mathfrak{n}}_C^-$ *spanned by arbitrary iterated Lie brackets of* F_i*'s*

$\widetilde{\mathfrak{h}}_C$ *spanned by the* H_i*'s*

Then we have

$$\widetilde{\mathfrak{g}}_C = \widetilde{\mathfrak{n}}_C^+ \oplus \widetilde{\mathfrak{h}}_C \oplus \widetilde{\mathfrak{n}}_C^-$$

with $\widetilde{\mathfrak{n}}_C^\pm$ *lying in degrees* $Q^\pm \backslash 0$ *and* $\widetilde{\mathfrak{h}}_C$ *in degree 0.*

Proof. Any element $x \in \widetilde{\mathfrak{g}}_C$ is a linear combination of iterated Lie brackets of E's, F's and H's. Let us consider any such Lie bracket, and reduce it via anti-symmetry and the Jacobi identity and relations (167)-(170), so that it has the minimal number of E's, F's and H's. We must show that if so reduced, then x must either be just an iterated Lie bracket of E's, or an iterated Lie bracket of F's, or a single H. Indeed, assume for the purpose of contradiction that some part of the iterated Lie bracket in question involved

$$\dots [F_j, [E_{i_1}, \dots, [E_{i_{k-1}}, E_{i_k}] \dots]] \dots$$

(the reason why we do not assume there are any H's instead of the E's in the formula above is that they could be readily simplified by (168)). Then by repeated applications of the Jacobi identity, we could ensure that the innermost Lie bracket is $[F_j, E_{i_a}]$ for some a, which can be simplified using (170). ☐

Proposition 29. $\widetilde{\mathfrak{n}}_C^+$ *and* $\widetilde{\mathfrak{n}}_C^-$ *are freely generated by* $\{E_i\}_{1 \leq i \leq r}$ *and* $\{F_i\}_{1 \leq i \leq r}$*, respectively, while*

$$\widetilde{\mathfrak{h}}_C = \bigoplus_{i=1}^r \mathbb{C} H_i$$

Proof. Let us consider the tensor algebra TV of the vector space $V = \bigoplus_{i=1}^r \mathbb{C} v_i$. There is an action

$$\widetilde{\mathfrak{g}}_C \curvearrowright TV$$

given by

$$E_i \cdot (v_{j_1} \otimes \dots \otimes j_n) = \sum_{1 \leq s \leq n \text{ s.t. } j_s = i} (c_{ij_{s+1}} + \dots + c_{ij_n})(v_{j_1} \otimes \dots \otimes v_{j_{s-1}} \otimes v_{j_{s+1}} \otimes \dots \otimes v_{j_n})$$

$$F_i \cdot (v_{j_1} \otimes \dots \otimes v_{j_n}) = v_i \otimes v_{j_1} \otimes \dots \otimes v_{j_n}$$

$$H_i \cdot (v_{j_1} \otimes \dots \otimes v_{j_n}) = -\sum_{s=1}^n (c_{ij_1} + \dots + c_{ij_n})(v_{j_1} \otimes \dots \otimes v_{j_n})$$

Check that the above action is well-defined, by verifying the Lie bracket relations (167)-(170). From the formula above, it is clear that the F_i do not satisfy any Lie algebra relations between themselves other than the ones prescribed by the free Lie algebra (by the very Definition 33). The analogous statement for the E_i is proved likewise. Finally, because

$$H_i \cdot v_j = -c_{ij} v_j$$

then any linear relation $\sum_{i=1}^{r} \gamma_i H_i = 0$ would imply

$$\sum_{i=1}^{r} \gamma_i c_{ij} = 0$$

for all j. This is impossible, as the Cartan matrix C has positive determinant. □

12.5

Let us now consider the ideals

$$\mathfrak{i}^{\pm} \subseteq \widetilde{\mathfrak{n}}_C^{\pm}$$

generated by relations (171) and (172), respectively.

Proposition 30. *The direct sum* $\mathfrak{i} = \mathfrak{i}^+ \oplus \mathfrak{i}^-$ *is an ideal in* $\widetilde{\mathfrak{g}}_C$*, and we have*

$$\mathfrak{g}_C = \widetilde{\mathfrak{g}}_C \big/ \mathfrak{i}$$

Proof. Let S_{ij}^+ and S_{ij}^- denote the LHS of (171) and (172), respectively. Prove the formulas

$$[F_k, S_{ij}^+] = 0, \quad \forall i, j, k \in \{1, \dots, r\} \tag{177}$$

$$[E_k, S_{ij}^-] = 0, \quad \forall i, j, k \in \{1, \dots, r\} \tag{178}$$

in $\widetilde{\mathfrak{g}}_C$ using repeated applications of the Jacobi identity and relations (168), (169), (170); if you prefer, you can work in the universal enveloping algebra by the injectivity of (90), where

$$S_{ij}^{\pm} = \sum_{k=0}^{1-c_{ij}} (-1)^k \binom{1-c_{ij}}{k} E_i^k E_j E_i^{1-c_{ij}-k}$$

By (177)-(178) and the fact that $\mathfrak{i}^{\pm}$ are ideals of $\widetilde{\mathfrak{n}}_C^{\pm}$, we conclude that $\mathfrak{i}^{\pm}$ are preserved under Lie bracket with all E's and F's. Because of (170) and the Jacobi identity, then $\mathfrak{i}^{\pm}$ are ideals of $\widetilde{\mathfrak{g}}_C$, and therefore so is their direct sum. We therefore obtain a surjective Lie algebra homomorphism

$$\widetilde{\mathfrak{g}}_C / \mathfrak{i} \twoheadrightarrow \mathfrak{g}_C \tag{179}$$

However, anything in the kernel of the above function would be a combination of iterated commutators of $S_{ij}^{\pm}$'s with E's and F's. By (177)-(178), any such commutator would already be in $\mathfrak{i}^{\pm}$, so (179) is an isomorphism. □

Proof. of Theorem 20: Let us start with an important observation: consider the adjoint action of any $\mathfrak{sl}_2$-triple E_i, F_i, H_i on $\mathfrak{g}_C$. The Serre relations (171)-(172) precisely imply that the subrepresentation generated by any E_j or F_j is finite-dimensional, with weights in $\{c_{ij}, \dots, -c_{ij}\}$. However, if the representations generated by x and y are finite-dimensional, then so is the representation generated by $[x, y]$ (specifically, it would be spanned by Lie brackets of the basis vectors of the aforementioned two representations). We conclude that any element $\mathfrak{g}_C$ generates a finite-dimensional subrepresentation with respect to any $\mathfrak{sl}_2$-triple. By Proposition 30, we have a decomposition

$$\mathfrak{g}_C = \mathfrak{h} \bigoplus_{\beta \in Q} \mathfrak{g}_{C,\beta}$$

with respect to the grading (176). By Proposition 28, all the direct summands above are finite-dimensional, and moreover

$$\mathfrak{g}_{C,\beta} = 0 \quad \text{if } \beta \notin Q^{\pm} \tag{180}$$

and

$$\mathfrak{g}_{C,k\alpha_i} = \begin{cases} \mathbb{C}E_i & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases} \quad \text{and} \quad \mathfrak{g}_{C,-k\alpha_i} = \begin{cases} \mathbb{C}F_i & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases} \tag{181}$$

We want to show that

$$\dim \mathfrak{g}_{C,\beta} = \begin{cases} 1 & \text{if } \beta \in R^{\pm} \\ 0 & \text{otherwise} \end{cases} \tag{182}$$

To this end, choose any $\beta \in Q$ and consider the adjoint action of the $\mathfrak{sl}_2$ triple E_i, F_i, H_i on

$$T_{\beta,i} = \bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}_{C,\beta+\ell\alpha_i} \tag{183}$$

The ℓ-th direct summand above has weight with respect to H_i equal to

$$\frac{2(\alpha_i, \beta)}{(\alpha_i, \alpha_i)} + 2\ell$$

As explained in the first paragraph of the proof, any vector of $T_{\beta,i}$ generates a finite-dimensional $\mathfrak{sl}_2$ representation. Therefore, Corollary 2 implies that $T_{\beta,i}$ is finite-dimensional. As a consequence, Corollary 1 implies that its subspaces of opposite weights have the same dimension, so in particular

$$\dim \mathfrak{g}_{C,\beta} = \dim \mathfrak{g}_{C,s_i(\beta)} \tag{184}$$

where s_i is the simple reflection corresponding to α_i. We leave it to you to show that if β is not a multiple of a root, then there exists a sequence of reflections s_i that will land it in $Q \backslash (Q^+ \cup Q^-)$; in this case (180) would imply the bottom option in (182). On the other hand if β is a multiple of a root, then the last sentence in Theorem 18 implies that there is a sequence of reflections s_i that will make it into a multiple of a simple root; in this case (181) would imply the top option in (182).

We showed that $\mathfrak{g}_C$ is finite-dimensional, and that the dimensions of its graded subspaces are given by (182). It remains to prove that $\mathfrak{g}_C$ is simple, and to this end consider a non-zero ideal $\mathfrak{i} \subset \mathfrak{g}_C$. Because the Cartan matrix is invertible, the operators $\{\mathrm{ad}_{H_i}\}_{i \in \{1,\dots,r\}}$ act with disjoint spectrum on the root spaces of $\mathfrak{g}_C$. Since $\mathfrak{i}$ is preserved by the aforementioned operators, then if some element of $\mathfrak{i}$ has a non-zero coefficient in some $\mathfrak{g}_{C,\alpha}$, then we can assume that $\mathfrak{i}$ contains the subspace $\mathfrak{g}_{C,\alpha}$ in question. By the same logic as in (184), this implies that $\mathfrak{i}$ contains $\mathfrak{g}_{C,s_i(\alpha)}$ for all i. Because the Weyl group acts transitively on the set of roots, then we conclude that $\mathfrak{i} = \mathfrak{g}_C$. □

Lecture 13

13.1

We will now use the root system R associated to a complex semisimple Lie algebra $\mathfrak{g}$ to describe its complex representations. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ as in Lecture 9, a choice of positive/negative roots $R = R^+ \sqcup R^-$, and write $\alpha_1, \dots, \alpha_r$ for the corresponding simple roots. Show that the following is an immediate consequence of Proposition 7, using the various $\mathfrak{sl}_2$-triples in $\mathfrak{g}$.

Proposition 31. *Any finite-dimensional representation $\mathfrak{g} \curvearrowright V$ has a **weight decomposition**, i.e.*

$$V = \bigoplus_{\lambda \in P} V_\lambda \tag{185}$$

*where its **weight subspaces** are*

$$V_\lambda = \left\{ v \in V \middle| x \cdot v = \lambda(x) v, \forall x \in \mathfrak{h} \right\} \tag{186}$$

and the direct sum in (185) *goes over the **(integral) weight lattice***

$$P = \left\{ \lambda \in \mathfrak{h}^* \middle| \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}, \forall i \in \{1, \dots, r\} \right\} \tag{187}$$

Note that by the very definition of a root system, the weight lattice contains the root lattice (175)

$$P \supseteq Q \tag{188}$$

The two lattices are in general not equal (in fact, the quotient P/Q is a finite group whose order is equal to the determinant of the Cartan matrix). It is also a fact that the integrality condition on λ from (187) is equivalent to the a priori stronger condition $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all roots α.

Example 8. *When $\mathfrak{g} = \mathfrak{sl}_n$, the weight lattice is*

$$P = \left\{ (k_1, \dots, k_n) \in \mathbb{C}^n \middle| k_1 + \dots + k_n = 0,\ k_i - k_{i+1} \in \mathbb{Z},\ \forall i \in \{1, \dots, n-1\} \right\} \tag{189}$$

while the root lattice is

$$Q = \left\{ (k_1, \dots, k_n) \in \mathbb{C}^n \middle| k_1 + \dots + k_n = 0,\ k_i \in \mathbb{Z},\ \forall i \in \{1, \dots, n\} \right\} \tag{190}$$

The fact that $|P/Q| = n$ comes about by noting that each k_i in (189) *must be congruent to $\frac{d}{n}$ modulo $\mathbb{Z}$, for one and the same value of $d \in \{0, \dots, n-1\}$.*

13.2

Consider now the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

By definition, $\mathfrak{h}$ preserves the weight subspaces V_λ of any representation $\mathfrak{g} \curvearrowright V$. Moreover, show by using the Jacobi identity that

$$\boxed{\mathfrak{g}_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}} \tag{191}$$

for all $\alpha \in R, \lambda \in P$. Associated to our choice of positive roots $R = R^+ \sqcup R^-$, we write

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$$

where $\mathfrak{n}^\pm = \oplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha$. We will write

$$\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h} \tag{192}$$

which is called a **Borel subalgebra** (Prove that $\mathfrak{b}$ is a solvable subalgebra of $\mathfrak{g}$; it is actually a Theorem of Borel-Morozov that it is a maximal solvable subalgebra).

Definition 35. *We say that $\lambda \in P$ is a* ***highest weight*** *for a representation $\mathfrak{g} \curvearrowright V$ if*

$$V_\lambda \neq 0 \quad \textit{and} \quad V_{\lambda+\alpha_1} = \cdots = V_{\lambda+\alpha_r} = 0$$

A highest weight vector of P will be some non-zero $v \in V_\lambda$ as above.

Because of (191), any highest weight vector v satisfies

$$\mathfrak{n}^+ \cdot v = 0 \tag{193}$$

If the highest weight of v is λ, then we have

$$h \cdot v = \lambda(h) v, \quad \forall h \in \mathfrak{h} \tag{194}$$

Since one can always find a highest weight in any finite-dimensional representation, it is elementary to obtain the following.

Proposition 32. *For semisimple $\mathfrak{g}$, any finite-dimensional irreducible representation $\mathfrak{g} \curvearrowright V$ is a* ***highest weight representation****, i.e. it is generated by a highest weight vector.*

We will use the notion of highest weights to classify irreducible representations. As we have seen from Theorem 16, this would completely characterize the representation theory of complex semisimple Lie algebras, since any such representation uniquely decomposes as a direct sum of irreducible representations (moreover, in the next lecture, we will learn how to use characters in order to determine which particular irreducibles show up in the decomposition of any given representation).

13.3

Since irreducible representations are generated by highest weight vectors (as per Proposition 32), the first step in constructing them is to construct the universal representation satisfying (193) and (194).

Definition 36. *The* ***Verma module*** *with highest weight λ is*

$$M(\lambda) = U\mathfrak{g} \bigotimes_{U\mathfrak{b}} \mathbb{C} \tag{195}$$

where the tensor product is defined with respect to

- *the injection* $U\mathfrak{b} \hookrightarrow U\mathfrak{g}$ *of universal enveloping algebras corresponding to* $\mathfrak{b} \hookrightarrow \mathfrak{g}$
- *the surjection* $U\mathfrak{b} \twoheadrightarrow \mathbb{C}$ *which sends* $\mathfrak{n}^+$ *to* 0 *and every* $h \in \mathfrak{h}$ *to* $\lambda(h)$*.*

As (195) *is an* $U\mathfrak{g}$ *module with respect to the left action, Subsection 6.1 implies that it is also a representation of* $\mathfrak{g}$*. Note that it is infinite-dimensional.*

Remark. *You may recognize* $M(\lambda)$ *as being the induced representation* $\mathrm{Ind}_{U\mathfrak{b}}^{U\mathfrak{g}}(\mathbb{C}_\lambda)$*, where* $\mathbb{C}_\lambda$ *is the one-dimensional representation of* $\mathfrak{b}$ *corresponding to* $\mathfrak{n}^+$ *acting by 0 and* $\mathfrak{h}$ *acting by the weight* λ*. Indeed, in Math 314 you studied induced representations of finite groups, but the situation for infinite-dimensional algebras such as* $U\mathfrak{g}$ *is analogous.*

If we let v_λ be a generator of the one-dimensional representation $\mathbb{C}_\lambda$, then any element of $M(\lambda)$ is of the form xv_λ for some $x \in U\mathfrak{g}$. However, because of the PBW Theorem 10, we have an isomorphism

$$U\mathfrak{g} = U\mathfrak{n}^- \otimes U\mathfrak{h} \otimes U\mathfrak{n}^+ = U\mathfrak{n}^- \otimes U\mathfrak{b}$$

Therefore, any element $x \in U\mathfrak{g}$ is a linear combination of products of elements from $U\mathfrak{n}^-$ and elements of $U\mathfrak{b}$. Since any element of $U\mathfrak{b}$ acts on v_λ by multiplying it with a constant, while elements of $U\mathfrak{n}^-$ act on V_λ freely, the assignment $xv_\lambda \to x$ yields a vector space isomorphism

$$\boxed{M(\lambda) \cong U\mathfrak{n}^-} \tag{196}$$

Moreover, the dimension of the weight spaces of the above isomorphism match up: any element $x_1 \dots x_n v_\lambda$ (for various $x_k \in \mathfrak{g}_{-\beta_k}$) lies in the $\lambda - \beta_1 - \dots - \beta_n$ weight subspace of $M(\lambda)$, by repeated applications of (191). Therefore, (196) and the PBW Theorem 10 tell us that

$$\dim M(\lambda)_\mu = \left|\left\{\text{unordered positive roots } \beta_1, \dots, \beta_n \text{ with sum } \lambda - \mu\right\}\right| \tag{197}$$

In particular, $M(\lambda)$ has finite-dimensional weight subspaces.

13.4

Let $L(\lambda)$ be any irreducible representation of $\mathfrak{g}$ generated by a vector of highest weight $\lambda \in \mathfrak{h}^*$, even infinite-dimensional. There exists a homomorphism of $\mathfrak{g}$ representations

$$\pi : M(\lambda) \twoheadrightarrow L(\lambda)$$

defined by sending v_λ to a highest weight vector of $L(\lambda)$ (check that this homomorphism is well-defined and surjective, using (193) and (194)). By definition, the kernel of π is a proper $\mathfrak{g}$ subrepresentation of $M(\lambda)$ that is graded by weights $\mathfrak{h}^*$, and the irreducibility of $L(\lambda)$ implies that it is a maximal such proper graded subrepresentation.

Proposition 33. *For semisimple* $\mathfrak{g}$*, there exists up to isomorphism a unique irreducible representation* $\mathfrak{g} \curvearrowright L(\lambda)$ *generated by a vector of highest weight* $\lambda \in \mathfrak{h}^*$*.*

The Proposition is an easy consequence of the fact that $M(\lambda)$ has a unique maximal graded subrepresentation (simply take the sum of all graded proper subrepresentations, which does not coincide with $M(\lambda)$, because it cannot contain v_λ). Therefore, we will refer to the representation $L(\lambda)$.

Theorem 21. *For semisimple $\mathfrak{g}$, $L(\lambda)$ is finite-dimensional if and only if λ lies in*

$$P^+ = \left\{ \lambda \in P \;\middle|\; \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}_{\geq 0}, \forall i \in \{1, \dots, r\} \right\} \tag{198}$$

Such weights are called ***dominant****.*

For $\mathfrak{g} = \mathfrak{sl}_n$, a weight as in (189) is dominant if and only if $k_i - k_{i+1} \in \mathbb{Z}_{\geq 0}$ for all $i \in \{1, \dots, n-1\}$.

Proof. of Theorem 21: Assume that $L(\lambda)$ is finite-dimensional. Take an $\mathfrak{sl}_2$-triple E_i, F_i, H_i corresponding to any $i \in \{1, \dots, r\}$. Then $L(\lambda)$ is a finite-dimensional representation with respect to this $\mathfrak{sl}_2$, with the highest weight vector of $L(\lambda)$ having weight

$$\frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$$

As we saw in Lecture 5, the numbers above must be non-negative integers in order to have a finite-dimensional representation of $\mathfrak{sl}_2$, so we conclude that $\lambda \in P^+$. Conversely, assume $\lambda \in P^+$. We will actually prove the following stronger claim on the weight subspaces of $L(\lambda)$

$$\dim L(\lambda)_\mu = \dim L(\lambda)_{w(\mu)} \tag{199}$$

for all $\mu \in P$ and $w \in W$. Indeed, because $L(\lambda)$ is a quotient of $M(\lambda)$, then its weight subspaces are finite-dimensional and only non-zero in the cone $\{\lambda - m_1\alpha_1 - \dots - m_r\alpha_r\}_{m_1,\dots,m_r \geq 0}$. If there existed such a non-zero root subspace with $m_1 + \dots + m_r$ arbitrarily large, then by applying formula (199) for the element $w \in W$ which sends the positive roots to negative roots (Corollary 7), then we would conclude the existence of a non-zero subspace with weight $w(\lambda) + m'_1\alpha_1 + \dots + m'_r\alpha_r$ for arbitrarily large $m'_1 + \dots + m'_r$. As this is impossible, the only option is for $L(\lambda)$ to only have finitely many non-zero weight subspaces, hence it must be finite-dimensional.

Let us now prove (199). Let us consider an $\mathfrak{sl}_2$-triple E_i, F_i, H_i for every $i \in \{1, \dots, r\}$, and define

$$V \subseteq L(\lambda)$$

to consist of all vectors v on which the E_i's and F_i's act **locally nilpotently**, i.e.

$$E_i^N v = F_i^N v = 0$$

for all $i \in \{1, \dots, r\}$ and for some $N \geq 0$ which may repend on i and v. Firstly, the highest weight vector v_λ lies in V because for all $i \in \{1, \dots, r\}$ we have

$$E_i v_\lambda = 0 \quad \text{and} \quad F_i^{k+1} v_\lambda = 0, \text{ where } k = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$$

(note that the second equality is non-trivial, but we leave it to you to show that $F_i^{k+1} v_\lambda$ is annihilated by all the E_j's, and thus would generate a proper highest weight subrepresentation of $L(\lambda)$ if it weren't zero). Secondly, V is preserved by the action of $\mathfrak{g}$, because for any $x \in \mathfrak{g}$ and any $i \in \{1, \dots, r\}$ we have the following equality in $U\mathfrak{g}$

$$[E_i^N, x] = \sum_{M=0}^{N-1} \binom{N}{M} \underbrace{[E_i, [E_i, \dots, [E_i, x] \dots]]}_{N-M \text{ copies of } E_i} E_i^M$$

and analogously for F_i. Thus, if a vector v is annihilated by large enough powers of every E_i, then so if xv because the iterated commutators in the equation above will all be 0 if $N - M$ is large enough. The remarks labeled firstly and secondly above imply that V is a subrepresentation of $L(\lambda)$, hence $V = L(\lambda)$ due to the latter's irreducibility. Then let us consider any weights μ and $s_i(\mu)$ and define the subrepresentation (with respect to the $\mathfrak{sl}_2$-triple E_i, F_i, H_i) generated by the corresponding weight subspaces

$$L(\lambda)_\mu \oplus \cdots \oplus L(\lambda)_{s_i(\mu)}$$

By the discussion above, this subrepresentation is finite-dimensional, with $L(\lambda)_\mu$ and $L(\lambda)_{s_i(\mu)}$ having H_i-weights

$$\frac{2(\mu, \alpha_i)}{(\alpha_i, \alpha_i)} \quad \text{and} \quad \frac{2(s_i(\mu), \alpha_i)}{(\alpha_i, \alpha_i)} = -\frac{2(\mu, \alpha_i)}{(\alpha_i, \alpha_i)}$$

By Corollary 1, $L(\lambda)_\mu$ and $L(\lambda)_{s_i(\mu)}$ must have the same dimension, which yields (199) for $w = s_i$. Since the simple reflections generate the Weyl group, then (199) holds for all w. ☐

Together with Theorem 16, we conclude the following.

Corollary 8. *For semisimple $\mathfrak{g}$, any finite-dimensional representation $\mathfrak{g} \curvearrowright V$ is isomorphic to*

$$V \cong L(\lambda_1) \oplus \cdots \oplus L(\lambda_k)$$

for $\lambda_1, \ldots, \lambda_k \in P^+$.

13.5

Motivated by Theorem 21, we have the following.

Definition 37. *The **fundamental weights** $\omega_1, \ldots, \omega_r$ are defined such that*

$$\frac{2(\omega_j, \alpha_i)}{(\alpha_i, \alpha_i)} = \delta_{ij} \tag{200}$$

for all $i, j \in \{1, \ldots, r\}$.

Fundamental weights form a $\mathbb{Z}_{\geq 0}$-basis of the cone of dominant weights, meaning that any dominant weight is of the form $n_1\omega_1 + \cdots + n_r\omega_r$ for some $n_1, \ldots, n_r \in \mathbb{Z}_{\geq 0}$. This has the following effect on the representation theory: show that the tensor product

$$\mathfrak{g} \curvearrowright L(\omega_1)^{\otimes n_1} \otimes \cdots \otimes L(\omega_r)^{\otimes n_r}$$

has highest weight $n_1\omega_1 + \cdots + n_r\omega_r = \lambda$. By Corollary 8, we have

$$L(\omega_1)^{\otimes n_1} \otimes \cdots \otimes L(\omega_r)^{\otimes n_r} \cong L(\lambda) \bigoplus_{\beta \in Q^+ \backslash 0} L(\lambda - \beta)^{\oplus \text{multiplicities}}$$

where the multiplicities above can be construed as a generalization of the Clebsch-Gordan rule (81). The formula above implies that $L(\lambda)$ can be recursively constructed (up to irreducible representations of the form $L(\lambda - \beta)$ with $\beta \in Q^+ \backslash 0$) from tensor products of the fundamental representations, i.e. the irreducible representations corresponding to the fundamental weights. It is in this sense that the fundamental representations "generate" the finite-dimensional representation theory of $\mathfrak{g}$.

Example 9. *Whereas the simple roots of* $\mathfrak{sl}_n$ *are* $\{e_i - e_{i+1}\}_{1 \leq i \leq n-1}$, *the fundamental weights are*

$$\omega_i = \frac{n-i}{n} \cdot (e_1 + \cdots + e_i) - \frac{i}{n} \cdot (e_{i+1} + \cdots + e_n) \tag{201}$$

The irreducible representation corresponding to ω_i *is none other than*

$$\wedge^i \mathbb{C}^n \tag{202}$$

Indeed, the highest weight vector of $\wedge^i \mathbb{C}^n$ *is* $v_1 \wedge \cdots \wedge v_i$, *which is an eigenvector for the action of any* $x = (x_1, \ldots, x_n) \in \mathfrak{h}$ *with eigenvalue*

$$x_1 + \cdots + x_i = (\omega_i, x)$$

To see that $\wedge^i \mathbb{C}^n$ *is irreducible, take any linear combination of tensors*

$$w = v_{t_1} \wedge \cdots \wedge v_{t_i} + \ldots$$

where $t_1 < \cdots < t_i$ *and the ellipsis stands for sequences lexicographically smaller than* $(t_1, \ldots, t_i)$. *Then applying the operators* $\{E_{dt_d}\}_{1 \leq d \leq i, t_d \neq d}$ *in succession to* w *would produce* $v_1 \wedge \cdots \wedge v_i$.

Lecture 14

14.1

In Math 314, you saw that characters are certain functions on a finite group that completely determine its representation theory. For semisimple Lie algebras $\mathfrak{g}$, the analogous role is taken by the following notion.

Definition 38. *The **character** of a representation $\mathfrak{g} \curvearrowright V$ is the sum*

$$\chi_V = \sum_{\lambda \in P} (\dim V_\lambda) e^\lambda \tag{203}$$

where $\{e^\lambda\}_{\lambda \in P}$ are formal symbols.

Although e^λ is a formal symbol, it arises from the following construction. As per Subsection 4.5, the representation $\mathfrak{g} \curvearrowright V$ lifts to a representation of the simply connected Lie group

$$G \curvearrowright V$$

with Lie algebra $\mathfrak{g}$ (this G is also called semisimple). There is an abelian subgroup called **maximal torus**

$$H \subset G$$

with Lie algebra $\mathfrak{h}$, and integral weights lift to characters

$$\lambda : H \to \mathbb{C}^*$$

Then we have for all $t = \exp(x) \in H$ (where $x \in \mathfrak{h}$ is arbitrary)

$$\chi_V(t) = \sum_{\lambda \in P} (\dim V_\lambda) e^{\lambda(x)} = \mathrm{tr}(t|_V) \tag{204}$$

This is now closer to the usual definition of characters as traces of group elements acting in the representation V. Of course, you may object that (204) only measures the trace on elements of H and not of G. But because the trace is conjugation invariant, the formula above actually measures the trace on any conjugates of H, which are dense in G (think about SL_n and arbitrary conjugates of diagonal matrices).

14.2

Another reason why we prefer formal expressions like (203) to actual numbers like (204) is that the former also applies to infinite-dimensional representations V (with finite-dimensional weight spaces) while the latter only applies to finite-dimensional representations. For example, (197) implies that

$$\chi_{M(\lambda)} = \frac{e^\lambda}{\prod_{\alpha \in R^+} (1 - e^{-\alpha})} = \frac{e^{\lambda + \rho}}{\prod_{\alpha \in R^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})} \tag{205}$$

(the reason for the shift in the numerator by e^ρ, where $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$, will be made apparent in Theorem 22). To make the above formula precise, we expand the denominator as a power series

$$\frac{1}{1 - e^{-\alpha}} = 1 + e^{-\alpha} + e^{-2\alpha} + \dots$$

and use the following operations on the formal symbols e^λ:

$$e^\lambda e^\mu = e^{\lambda+\mu}$$

$$\overline{e^\lambda} = e^{-\lambda}$$

The motivation for these operations is given by the following formulas, which we invite you to prove

$$\chi_{V\oplus V'} = \chi_V + \chi_{V'} \tag{206}$$

$$\chi_{V\otimes V'} = \chi_V \chi_{V'} \tag{207}$$

$$\chi_{V^\vee} = \overline{\chi_V} \tag{208}$$

with respect to direct sums, tensor products and dual representations (see (37), (38), (39))

14.3

We will now calculate the character of irreducible representations $L(\lambda)$ of a semisimple Lie algebra $\mathfrak{g}$. The key result is the following formula of Freudenthal, which allows one to recursively compute the dimensions of the weight spaces of any irreducible representation starting from the obvious

$$\dim L(\lambda)_\lambda = 1$$

Proposition 34. *For any $\mu \in P$, we have*

$$\Big((\lambda+\rho,\lambda+\rho) - (\mu+\rho,\mu+\rho)\Big)\dim L(\lambda)_\mu = 2\sum_{\alpha\in R^+}\sum_{k=1}^{\infty}(\mu+k\alpha,\alpha)\dim L(\lambda)_{\mu+k\alpha} \tag{209}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$ (note that the sum in the RHS is actually finite).

Proof. Let us consider the Casimir element associated to the non-degenerate s.i.b.f. of $\mathfrak{g}$

$$C = \sum_{i=1}^{r} H_i H^i + \sum_{\alpha\in R^+}\frac{(\alpha,\alpha)}{2}(E_\alpha F_\alpha + F_\alpha E_\alpha)$$

where H_i and H^i are dual bases of $\mathfrak{h}$ (the latter basis can be readily expressed in terms of the former basis using the formulas $(H_\alpha, H_\beta) = \frac{4(\alpha,\beta)}{(\alpha,\alpha)(\beta,\beta)}$, but we will not need this).

Lemma 9. *C acts on $L(\lambda)$ via the scalar $(\lambda+\rho,\lambda+\rho)-(\rho,\rho)$.*

Proof. Since C is central, we know that it acts on $L(\lambda)$ as a scalar, so it remains to identify this scalar by calculating how C acts on the highest weight vector v_λ. Any $\mathfrak{sl}_2$-triple $E_\alpha, F_\alpha, H_\alpha$ will have the property that

$$E_\alpha v_\lambda = 0 \quad \Rightarrow \quad F_\alpha E_\alpha v_\lambda = 0$$

hence

$$E_\alpha F_\alpha v_\lambda = H_\alpha v_\lambda = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} v_\lambda$$

On the other hand, it is an easy manipulation with symmetric bilinear forms that

$$\sum_{i=1}^{r} H_i H^i v_\lambda = (\lambda,\lambda) v_\lambda \tag{210}$$

Since $(\lambda,\lambda) + \sum_{\alpha\in R^+}(\lambda,\alpha) = (\lambda+\rho,\lambda+\rho)-(\rho,\rho)$, the Lemma follows. □

Let us now consider any positive root $\alpha \in R^+$, and decompose $L(\lambda)$ into subrepresentations of the $\mathfrak{sl}_2$-triple $E_\alpha, F_\alpha, H_\alpha$. These will all be of the form

$$L(\lambda)_{\beta+\frac{m}{2}\alpha} \oplus L(\lambda)_{\beta+\frac{m-2}{2}\alpha} \oplus \cdots \oplus L(\lambda)_{\beta-\frac{m-2}{2}\alpha} \oplus L(\lambda)_{\beta-\frac{m}{2}\alpha} \tag{211}$$

where we assume that m is maximal such that the above weight spaces actually appear in $L(\lambda)$. The shift by β is chosen so that $(\alpha, \beta) = 0$, which implies that as a representation of the $\mathfrak{sl}_2$-triple $E_\alpha, F_\alpha, H_\alpha$, the $\beta + \frac{d}{2}\alpha$ direct summand above has weight d. By the inclusion-exclusion principle, the number of copies of the irreducible representation $\mathfrak{sl}_2 \curvearrowright L(n)$ in the representation (211) is

$$\dim L(\lambda)_{\beta+\frac{n}{2}\alpha} - \dim L(\lambda)_{\beta+\frac{n+2}{2}\alpha}$$

for all $n \geq 0$. Meanwhile, (74) and (75) imply that $E_\alpha F_\alpha + F_\alpha E_\alpha$ acts on the d-th weight subspace of an irreducible representation $\mathfrak{sl}_2 \curvearrowright L(n)$ by the constant

$$\frac{n(n+2)}{2} - \frac{d^2}{2}$$

Let's assume $d \geq 0$ for simplicity. We conclude that $E_\alpha F_\alpha + F_\alpha E_\alpha$ acts on $L(\lambda)_{\beta+\frac{d}{2}\alpha}$ with trace

$$\sum_{n=d}^{\infty} \frac{n(n+2)}{2} \Big(\dim L(\lambda)_{\beta+\frac{n}{2}\alpha} - \dim L(\lambda)_{\beta+\frac{n+2}{2}\alpha} \Big) - \frac{d^2}{2} \dim L(\lambda)_{\beta+\frac{d}{2}\alpha}$$

If we write $\mu = \beta + \frac{d}{2}\alpha \Leftrightarrow d = \frac{2(\mu,\alpha)}{(\alpha,\alpha)}$ and manipulate the telescoping sum above, we conclude that

$$\frac{(\alpha,\alpha)}{2}(E_\alpha F_\alpha + F_\alpha E_\alpha)$$

acts on $L(\lambda)_\mu$ with trace

$$(\mu, \alpha) \dim L(\lambda)_\mu + 2 \sum_{k=1}^{\infty} (\mu + k\alpha, \alpha) \dim L(\lambda)_{\mu+k\alpha}$$

Summing over all $\alpha \in R^+$ and adding to the mix the fact (analogous to (210)) that $\sum_{i=1}^r H_i H^i$ acts in $L(\lambda)_\mu$ as multiplication with (μ, μ), we conclude that C acts on $L(\lambda)_\mu$ with trace

$$2(\mu, \rho) \dim L(\lambda)_\mu + 2 \sum_{\alpha \in R^+} \sum_{k=1}^{\infty} (\mu + k\alpha, \alpha) \dim L(\lambda)_{\mu+k\alpha}$$

Comparing this with Lemma 9 implies (209). □

14.4

A purely algebraic manipulation (which you may find in §25.2 of the book by Fulton-Harris) allows one to deduce from Freudenthal's formula (209) the following so-called **Weyl character formula**.

Theorem 22. *The character of any irreducible representation $\mathfrak{g} \curvearrowright L(\lambda)$ is given by the formula*

$$\chi_{L(\lambda)} = \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)}}{\prod_{\alpha \in R^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})} \tag{212}$$

where $\operatorname{sgn} : W \to \{\pm 1\}$ *is the group homomorphism that sends the simple reflections* s_i *to* -1.

Certain observations about (212) are in order.

1. The $w = e$ summand in the numerator of (212) yields precisely the character of the Verma module in (205), and this corresponds to the fact that $M(\lambda)$ contains a copy of the irreducible representation $L(\lambda)$ generated by the highest weight vector. Conversely, we can interpret the right-hand side of (212) as an alternating sum of the right-hand sides of (205). This underlies the famous BGG (Bernstein-Gelfand-Gelfand) resolution of $L(\lambda)$ as a complex of Verma modules.

2. The numerator of (212) is an antisymmetric expression with respect to the Weyl group action, i.e. the operation $\{e^{\mu} \rightsquigarrow e^{w(\mu)}\}_{w \in W}$, has the effect of multiplying the numerator of (212) by $\mathrm{sgn}(w)$. It is a general property of Coxeter groups that the denominator of (212) is also antisymmetric, and it divides the numerator. Therefore, we conclude that the right-hand side of (212) is a linear combination of e^{μ}'s, which is symmetric with respect to the W-action (as expected from (199)).

3. We can get a formula for the dimension of $L(\lambda)$ by taking the evaluation of the right-hand side of (212) as $e^{\mu} \rightsquigarrow e^{\mu(0)}$ where 0 is the origin of $\mathfrak{h}$. This is strictly speaking ill-defined, since we get $\frac{0}{0}$. The way to resolve this issue is to evaluate the right-hand side of (212) as $e^{\mu} \rightsquigarrow e^{\mu(x)}$ for $x \in \mathfrak{h}$ tending to 0 along a generic line. If you do so appropriately, you will find that

$$\dim L(\lambda) = \prod_{\alpha \in R^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)} \tag{213}$$

14.5

Finally, let us consider the whole discussion above in the particularly important case $\mathfrak{g} = \mathfrak{sl}_n$. For a weight $\lambda = (k_1, \dots, k_n)$ with $k_1 + \dots + k_n = 0$, we will write

$$e^{\lambda} = z_1^{k_1} \dots z_n^{k_n}$$

The Weyl group $W = S_n$ acts on the monomials above by permuting the variables $z_1, \dots, z_n$, and $\mathrm{sgn} : S_n \to \{\pm 1\}$ is the usual sign homomorphism. For any root $\alpha = e_i - e_j$, we have

$$e^{\alpha} = \frac{z_i}{z_j}$$

and $\rho = \frac{1}{2}(n-1, n-3, \dots, 3-n, 1-n)$. Therefore, the Weyl character formula reads (after some minor algebraic manipulations, which we invite you to do)

$$\chi_{L(\lambda)} = \frac{\sum_{w \in S_n} \mathrm{sgn}(w) \prod_{i=1}^{n} z_{w(i)}^{k_i + n - i}}{\prod_{1 \le i < j \le n} (z_i - z_j)}$$

One recognizes the denominator of the right-hand side as the Vandermonde determinant, so

$$\chi_{L(\lambda)} = \frac{\begin{vmatrix} z_1^{k_1+n-1} & z_2^{k_1+n-1} & \dots & z_n^{k_1+n-1} \\ z_1^{k_2+n-2} & z_2^{k_2+n-2} & \dots & z_n^{k_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{k_n} & z_2^{k_n} & \dots & z_n^{k_n} \end{vmatrix}}{\begin{vmatrix} z_1^{n-1} & z_2^{n-1} & \dots & z_n^{n-1} \\ z_1^{n-2} & z_2^{n-2} & \dots & z_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{0} & z_2^{0} & \dots & z_n^{0} \end{vmatrix}} \tag{214}$$

As we saw in Example 8, a dominant weight of $\mathfrak{sl}_n$ has the property that

$$k_i = \ell_i - \frac{\ell_1 + \dots + \ell_n}{n}, \quad \forall i \in \{1, \dots, n\}$$

for some partition of non-negative integers

$$\ell = (\ell_1 \geq \ell_2 \geq \dots \geq \ell_n \geq 0)$$

Then the right-hand side of (214) is equal to

$$\frac{s_\ell(z_1, \dots, z_n)}{(z_1 \dots z_n)^{\frac{\ell_1 + \dots + \ell_n}{n}}}$$

where s_ℓ is the famous Schur polynomial associated to the partition ℓ. For example, when λ is the fundamental weight (201), we have

$$\ell = (1 \geq \dots \geq 1 \geq 0 \geq \dots \geq 0)$$

with i ones and $n - i$ zeroes. The corresponding Schur polynomial is none other than the i-th elementary symmetric polynomial, so (214) states that

$$\chi_{\wedge^i \mathbb{C}^n} = \frac{\sum_{1 \leq t_1 < \dots < t_i \leq n} z_{t_1} \dots z_{t_i}}{(z_1 \dots z_n)^{\frac{i}{n}}}$$

which we invite you to prove is indeed the character of the i-th exterior power of the standard representation of $\mathfrak{sl}_n$.

If you recognize Schur polynomials as the characters of irreducible representations of the symmetric group from Math 314, this is no accident. An important result called Schur-Weyl duality shows that there exist commuting actions of $\mathfrak{gl}_n$ and S_N on

$$\underbrace{\mathbb{C}^n \otimes \dots \otimes \mathbb{C}^n}_{N \text{ factors}}$$

which provide a bridge between irreducible representations of $\mathfrak{gl}_n$ (which are very closely related to those of $\mathfrak{sl}_n$ that we just studied) and irreducible representations of S_N.